Associated Class Functions and Characteristic Polynomials on the Symmetric Group

Dissertation

zur

Erlangung der naturwissenschaftlichen Doktorwürde
(Dr.sc.nat)

vorgelegt der

Mathematisch-naturwissenschaftlichen Fakultät

der

Universität Zürich

von

Dirk Zeindler

von

Zürich ZH

Promotionskomitee

Prof. Dr. Ashkan Nikeghbali (Vorsitz)
Dr. Christopher Hughes (York)

Zürich 2010

Abstract

The characteristic polynomial of random matrices has been of great interest in recent years in many areas of mathematics and physics. Many papers in this direction deal with compact Lie groups, but there are almost no papers about discrete groups. This was one of our motivation to take a look at the group of permutation matrices, which is isomorphic to S_n. Another reason is that the symmetric group is a well studied object and many results about S_n are well known.
The topic of this thesis is the asymptotic behavior of two sort of class functions on S_n: the characteristic polynomial $Z_n(x)$ of permutation matrices and associated class functions $W^n(f)(x)$, which are a generalization of the characteristic polynomial. We have introduced them since we have realized that many calculations for $Z_n(x)$ also work for $W^n(f)(x)$.
We begin by proving a central limit theorem for $W^n(f)(x)$. The idea is to adapt the proof of a central limit theorem for $Z_n(x)$ used in the paper *The characteristic polynomial of a random permutation matrix.*
We then look at the moments $\mathbb{E}_{\mathcal{S}_n}\left[Z_n^s(x)\right]$. We use representation theory to write down the generating function and use it to calculate the asymptotic behavior. We distinguish several cases: $(|x| < 1, s \in \mathbb{N})$, $(|x| < 1, s \in \mathbb{C})$, $(|x| = 1, s = 1)$ and $(|x| = 1, s = 2)$.
We use as next a combinatorial argument to write down the generating function of $\mathbb{E}_{\mathcal{S}_n}\left[W^n(f)(x)\right]$. As previously we use it to calculate the asymptotic behavior for $n \to \infty$. We distinguish between the cases when f is a polynomial and when f is a holomorphic function.
Finally we take a short look at the characteristic polynomial on the alternating group and some Weyl groups.

Zusammenfassung

In den letzten Jahren stiess das charakteristische Polynom einer Zufallsmatrix in vielen Bereichen der Mathematik und Physik auf grosses Interesse. Viele Veröffentlichungen darüber behandeln kompakte Lie Gruppen, aber nur wenige diskrete Gruppen. Dies war einer der Gründe die Gruppe der Permutationsmatrizen zu untersuchen. Ein weiterer Grund ist das die Gruppe der Permutationsmatrizen isomorph zu symmetrischen Gruppe S_n ist, über welche schon seit geraumer Zeit sehr viel bekannt ist.
Das Thema dieser Dissertation das asymptotische Verhalten von zwei Sorten von Klassenfunktionen auf S_n: Das charakteristische Polynom $Z_n(x)$ einer Permutation Matrix und assoziierten Klassenfunktionen $W^n(f)(x)$, welche eine Verallgemeinerung des charakteristischen Polynoms einer Permutationsmatrix sind. Wir haben diese eingeführt da die meisten Berechnungen für $Z_n(x)$ auch für $W^n(f)(x)$ funktionieren.
Als erstes beweisen wir einen zentralen Grenzwertsatz für $W^n(f)(x)$. Die Beweisidee stammt aus dem Paper *The characteristic polynomial of a random permutation matrix*, wo ein zentraler Grenzwertsatz für $Z_n(x)$ bewiesen wird.
Als nächstes benutzen wir Darstellungstheorie um eine erzeugende Funktion für die Momente $\mathbb{E}_{\mathcal{S}_n}[Z_n^s(x)]$ herzuleiten. Wir benutzen dann diese erzeugende Funktion um das asymptotische Verhalten von $\mathbb{E}_{\mathcal{S}_n}[Z_n^s(x)]$ in den Fällen $(|x| < 1, s \in \mathbb{N})$, $(|x| < 1, s \in \mathbb{C})$ und $(|x| = 1, s \leq 2)$ zu berechnen.
Wir betrachten dann $\mathbb{E}_{\mathcal{S}_n}[W^n(f)(x)]$ und benutzen kombinatorische Mittel um eine erzeugende Funktion zu berechnen. Wie zuvor benutzen wir diese erzeugende Funktion um das asymptotische Verhalten von $\mathbb{E}_{\mathcal{S}_n}[W^n(f)(x)]$ herzuleiten. Wir unterscheiden dabei die Fälle f ein Polynom und f eine holomorphe Funktion.
Als letztes Betrachten wir noch kurz das charakteristische Polynom über der alternierenden Gruppe A_n sowie über einigen Weyl-Gruppen.

Acknowledgments

I want to thank all who have helped me with my dissertation. Especially I would like to thank my family, Prof. Ashkan Nikeghbali and Dr. Paul-Olivier Dehaye.

Contents

Chapter 1

Introduction

The study of random matrices has gained importance in many areas of mathematics and physics, for example in nuclear physics, infinite dimensional integrable systems and large$-n$ representation theory. Random matrix theory (RMT) was in the recent years also of great interest in number theory since the study of the spectrum of the characteristic polynomial of a random matrix in a compact Lie group was central in obtaining conjectures about the Riemann zeta function and families of $L-$functions (see, for instance, the book [23] and many papers in its reference list).

1.1 RMT and the Riemann zeta function $\zeta(s)$

1.1.1 Random matrix theory (RMT)

Random matrix theory is essentially the probabilistic study of various ensembles of matrices. (An ensemble is a set with an attached probability measure). We work only on the symmetric group S_n, which is isomorphic to a subgroup of unitary group $U(n)$ (see section 2.1.4). We state here for completeness some of the other ensembles.

1.1.1.1 Compact groups

On a compact Lie group G there exists a (unique) $G-$invariant measure $\mathbb{P}$, i.e. $\mathbb{P}[gA] = \mathbb{P}[A]$ for all $g \in G, A$ open (see [8]). This measure is called the Haar measure. All the classical compact groups as $U(n), SO(n), \cdots$ are Lie groups and can be therefore endowed with the Haar-measure. In fact all compact Lie groups can be embedded into the unitary group $U(N)$ for N large enough and therefore considered as matrices. This follows from the theorem of Peter and Weyl (see [8], section III.4). Unfortunately it is not so easy to work directly with the Haar-measure since it is obviously difficult to get good charts. Here comes in Weyl's integration formula (see [8], section IV.1), which reformulates the expectation $\mathbb{E}_G[f]$ of a class function f as an integral over a domain of the form $[0,1]^N$. A good example to illustrate this is the unitary group. We have

$$\mathbb{E}_{U(n)}[f] = \frac{1}{n!}\int_{[0,1]^n} f\big(\mathrm{diag}(e^{2\pi i\theta_1}, \cdots, e^{2\pi i\theta_n})\big) \prod_{1\le j<k\le n} \left|e^{2\pi i\theta_j} - e^{2\pi i\theta_k}\right|^2 \prod_{j=1}^{n} d\theta_j \quad (1.1.1)$$

with $\mathrm{diag}(d_1, \cdots, d_n)$ the diagonal matrix with diagonal entries $d_1, \cdots, d_n$. Similar formulas holds for other classical groups.

1.1.1.2 Other ensembles

Of course it is not necessary that the matrices form a multiplicative group. An example is the Gaussian unitary ensemble (GUE). This is the set of all $n \times n$ hermitian matrices with a probability measure that is invariant under $H \to U^{-1}HU$ and with the real and imaginary part of the matrix elements on and above the diagonal iid random variables. Other examples are the Gaussian Orthogonal Ensemble (GOE) or the real symmetric Wigner matrix. We refer here to the paper [12].

1.1.2 The Riemann zeta function and RMT

The Riemann zeta function is defined for $\mathrm{Re}(s) > 1$ as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1.1.2}$$

Riemann showed 1859 in his paper [7] that $\zeta(s)$ can be continued meormorphically to the whole complex plan and that ζ fulfils

$$\zeta(s) = \chi(s)\zeta(1-s) \tag{1.1.3}$$

with

$$\chi(s) = \pi^{s-1/2} \frac{\Gamma\left(\frac{1}{2}(1-s)\right)}{\Gamma\left(\frac{1}{2}s\right)}. \tag{1.1.4}$$

Riemann used this equation to show that $\zeta(s)$ has a simple pole at $s = 1$ and simple zeros at the negative even integers and there are no other zeros and poles for $\mathrm{Re}(s) > 1$ and for $\mathrm{Re}(s) < 0$. Riemann also showed that ζ has infinitely many zeros in the strip $0 \leq \mathrm{Re}(s) \leq 1$. This observation and the symmetry in (1.1.3) is the origin of Riemann's famous conjecture. He conjectured that all zeros of ζ in the critical strip $0 \leq \mathrm{Re}(s) \leq 1$ must have $\mathrm{Re}(s) = \frac{1}{2}$. This is still an open question. This conjecture is the reason why the behavior of ζ on the critical line has been studied by many mathematicians. One results in this direction is

$$\lim_{T\to\infty} \frac{1}{T} \mathbb{P}\left[T \leq t \leq 2T; \frac{\log\left(\zeta(\frac{1}{2}+it)\right)}{\sqrt{\frac{1}{2}\log\log T}} \in B\right] = \frac{1}{2\pi} \int\int_B e^{-(x^2+y^2)/2}\, dxdy. \tag{1.1.5}$$

with $\mathbb{P}[\cdot]$ the Lebesgue measure. This theorem was proven by Selberg (see [21]).
Modern approaches to the Riemann zeta function is through RMT. One of the reasons is Montgomery's pair correlation conjecture. Montgomery claimed that the pair correlation between pairs of zeros of the Riemann zeta function (normalized to have unit average spacing) is the same as the pair correlation function of random Hermitian matrices. Another connection between ζ and RMT appears in the "*Random matrix theory and* $\zeta(1/2+it)$" of Keating and Snaith [19]. They first compute the moments of the characteristic polynomial

$$\int_{U(n)} |\det(I-g)|^s\, dg = \prod_{j=1}^{n} \frac{\Gamma(j)\Gamma(j+s)}{(\Gamma(j+s/2))^2}, \tag{1.1.6}$$

$$\int_{U(n)} \left(\frac{\det(I-g)}{\det(I-g^*)}\right)^{s/2} dg = \prod_{j=1}^{n} \frac{\Gamma^2(j)}{\Gamma(j+s/2)\Gamma(j-s/2)} \tag{1.1.7}$$

and use this to show that

Theorem 1.1.1. *Let x be a fixed complex number with $|x| = 1$ and g_n be a unitary matrix chosen at random with respect to the Haar measure. Then*

$$\frac{\mathrm{Log}\Big(\det(I_n - xg_n)\Big)}{\sqrt{\frac{1}{2}\log(n)}} \xrightarrow{d} \mathcal{N}_1 + i\mathcal{N}_2 \text{ for } n \to \infty$$

and $\mathcal{N}_1, \mathcal{N}_2$ independent, normal distributed random variables.

Since this theorem is similar to (1.1.5), they compared the asymptotic behavior of the moments of $\det(I-g)$ with some numerical calculations for the moments of ζ on the critical line of Odlyzko (see [1]). This leads to the Keating and Snaith conjecture for the moments of ζ.

We do not want to give here further details on the Riemann zeta function and the characteristic polynomial of the unitary group since neither is a topic in this thesis. A good overview can be found is the thesis "*On the Characteristic Polynomial of a Random Unitary Matrix and the Riemann Zeta Function*" of C.Hughes [18].

1.2 Some related work on the symmetric group

The topic of this thesis is the symmetric group S_n which is isomorphic to a subgroup of unitary group $U(n)$ (see section 2.1.4). A surprising fact, proven in [17], is that there is a central limit theorem similar to theorem 1.1.1 for the characteristic polynomial on S_n. We give the precise formulation in section 3, theorem 3.1.2.
Also the distribution of the eigenvalues has been of interest. K.Wieand showed in her paper "*Eigenvalue distributions of random permutation matrices*" [25] that

$$Y_n^I = \frac{X_n^I - \mathbb{E}_{\mathcal{S}_n}\left[X_n^I\right]}{(c_2 \log(n))^{1/2}} \xrightarrow{d} \mathcal{N}(0,1) \tag{1.2.1}$$

where $X_n^I = X_n^I(\sigma)$ is the number of eigenvalues of $\sigma \in S_n$ in the interval $I = (e^{2\pi i\alpha}, e^{2\pi i\beta}]$. She showed in [26] that (1.2.1) is also true for the ensemble constructed from S_n by replacing the nonzero entries of each σ with a $M \times M$ diagonal matrices whose entries are random $K-th$ roots of unity or random points on the unit circle.

1.3 Overview of this thesis

The main topic of this thesis is the asymptotic behavior of certain class functions on symmetric group S_n. One of the motivations for this thesis is that many calculations and proofs on $U(n)$ are very difficult. A good example to illustrate this is Theorem 1.1.1. It was first shown by Constin and Lebowitz in [10] that the imaginary part of $\frac{\mathrm{Log}\left(\det(I_n - x g_n)\right)}{\sqrt{\frac{1}{2}\log(n)}}$ converges in distribution to a normal distributed random variable. They conjectured that the same is true for the real part and that the imaginary part and the real part are independent in the limit. It then needs about 10 years before Keating and Snaith have been able to prove this.
The idea to simplify to calculations is the following: Before looking at the whole unitary group $U(n)$, one does the computations for a simple subgroup and then try to transfer the methods to $U(n)$. This thesis shows that the subgroup that S_n is a good candidate for such a subgroup. We prove in section 3 that there is a central limit theorem for $Z_n(x)$ on S_n similar to Theorem 1.1.1. This result is an extension of a central limit theorem for $Z_n(x)$ proven in [17]. The difference between our result and the result in [17] is that we show the independence of the real and the imaginary part. We consider there also more general class functions on S_n.
Since S_n is a only a subgroup of $U(n)$, there are of course some difference in the methods and the results. For example, one can use representation theory on both groups, but one can use on S_n also combinatorial methods. An other difference is in the computation of

the moments. We will see in section 4 that we can not give a (non trivial) expression for the moments of $\mathbb{E}_{\mathcal{S}_n}\left[\prod_{l=1}^{L} Z_n(x_l)\right]$ but we can calculate the limit for $n \to \infty$. On $U(n)$, One can give explicit expressions for $\mathbb{E}_{U(n)}[\prod_{l=1}^{L} \det^s(I - x_l g)]$, but has often problems calculating the limit.

The structure of this thesis is as follows: We present in section 2 some well known facts from many directions of mathematics, which we need in the rest of this thesis .
In section 3 we introduce the characteristic polynomial $Z_n(x)$ and associated class function $W^n(f)(x)$ on S_n and then prove a central limit theorem for both.
In section 4 we use representation theory to write down the generating function of $\mathbb{E}_{\mathcal{S}_n}\left[Z_n^s(x)\right]$ and use it to obtain some asymptotics.
In section 5 we write down with a combinatorial argument the generating function of $\mathbb{E}_{\mathcal{S}_n}\left[W^n(f)(x)\right]$ and use it to get some asymptotics. We also introduce some random shifts of eigenvalues.
We finally present in a short section 6 the characteristic polynomial on the alternating group A_n and some Weyl groups.

Chapter 2

Basics

This section is dedicated to introduce some basics, which we need in the rest of this thesis. Most of what we present in this section is well known. The reader who is familiar with the matter can therefore skip the most part of this section. We recommend nevertheless to take a look at the Feller-coupling (section 2.1.3) and at the examples in section 2.7.

2.1 The symmetric group S_n

The main topic of this thesis is the symmetric group S_n. The symmetric group S_n appears in many areas of mathematics such as Galois theory, representation theory of Lie groups and combinatorics.
We do the following with the symmetric group: we integrate some class functions on S_n with respect to the uniform measure and take a look at the asymptotic behavior for $n \to \infty$.

For completeness, we state here the definition of the symmetric group.

Definition 2.1.1. *The* symmetric group $\mathcal{S}_n$ *is defined to be the set of all permutations of the set* $\{1, 2, \cdots, n\}$ *with the usual group operation. We write* $\epsilon(\sigma)$ *for the signature of a permutation* $\sigma \in S_n$.

2.1.1 Conjugation classes of S_n and $\mathbb{E}[.]$

The material of this subsection is well known and can be found for instance in [22]. We therefore give only a short overview and omit most of the proofs.
One of the things we look at is $\mathbb{E}_{S_n}[u]$ for u a class function (i.e. $u(hgh^{-1}) = u(g)$) and with respect to the uniform measure on S_n. We use in section 2.7 representation theory to give an expression for $\mathbb{E}_{S_n}[u]$ for some characters. In section 5 we need a more general expression for $\mathbb{E}_{S_n}[u]$. We have by definition

$$\mathbb{E}_{S_n}[u] = \frac{1}{n!} \sum_{g \in S_n} u(g).$$

We know that u is constant on each conjugation class. Therefore it is natural rewrite the sum in (2.1.1) as sum over the conjugation classes.
We first parameterize the conjugation classes of S_n with partitions of n.

Definition 2.1.2. *A* partition λ *is a sequence of nonnegative integers* $\lambda_1 \geq \lambda_2 \geq \cdots$ *eventually trailing to 0s, which we usually omit. We use the notation* $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$. *The* length *of* λ *is the largest* l *such that* $\lambda_l \neq 0$. *We define the* size $|\lambda| := \sum_i \lambda_i$ *and we set for* $n \in \mathbb{N}$

$$\lambda \vdash n := \{\lambda \text{ partition } ; |\lambda| = n\}.$$

Remark: The explicit notation for partitions and cycles of S_n are very similar. It will be always clear form the context if we have a partition or cycle. Therefore there is no danger of confusion.

Let $\sigma \in \mathcal{S}_n$ be arbitrary. We can write $\sigma = \sigma_1 \cdots \sigma_l$ with σ_i disjoint cycles of length λ_i. Since disjoint cycles commute, we can assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$. We call the partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ the *cycle-type* of σ and write $\mathcal{C}_\lambda$ for the subset of $\mathcal{S}_n$ with cycle-type $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$.

Example 2.1.3. The cycle-type of $(123)(45)$ is $(3, 2)$.

We now state an important lemma

Lemma 2.1.4. *Two elements $\sigma, \tau \in \mathcal{S}_n$ are conjugate if and only if σ and τ have the same cycle-type. The sets $\mathcal{C}_\lambda$ with $|\lambda| = n$ are the conjugation classes of $\mathcal{S}_n$*

Proof. It is clear that the second part follows immediately form the first part. The first part follows from the following calculation. Let $\sigma \in S_n$ be arbitrary. It follows with a straight forward calculation that

$$\mu\big(a_1 a_2 a_3 \cdots a_l\big)\mu^{-1} = \big(\mu(a_1)\mu(a_2)\mu(a_3) \cdots \mu(a_l)\big). \tag{2.1.1}$$

where $\big(a_1 a_2 a_3 \cdots a_l\big)$ means the cycle that sends $a_i \to a_{i+1}$. Therefore the cycle-type is invariant under conjugation. If σ, τ have the same cycle-type then it is easy to construct a $\mu \in S_n$ with $\mu\tau\mu^{-1} = \sigma$.

□

We also need the cardinality of each $\mathcal{C}_\lambda$. We have

Lemma 2.1.5.

$$|\mathcal{C}_\lambda| = \frac{|S_n|}{z_\lambda} \text{ with } z_\lambda := \prod_{m=1}^{n} m^{c_m} c_m! \text{ and } c_m = c_m(\lambda) = \#\{\lambda_i; \lambda_i = m\}. \tag{2.1.2}$$

Proof. The main idea is to calculate the cardinality of the set $\big\{\sigma \in S_n; \sigma\mathcal{C}_\lambda\sigma^{-1} = \mathcal{C}_\lambda\big\}$. More details can be found in [22] or [9]. □

We put everything together and get

Lemma 2.1.6. *Let $u : \mathcal{S}_n \to \mathbb{C}$ be a class function. Then*

$$\mathbb{E}_{S_n}[u] = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} u(\lambda). \tag{2.1.3}$$

2.1.2 Cycle counts and asymptotic behavior

Definition 2.1.7. *Let $\sigma \in S_n$ be given with cycle-type $\lambda = (\lambda_1, \cdots, \lambda_l)$. We define*

$$C_m := C_m^{(n)} := C_m^{(n)}(\sigma) := \#\{i | 1 \leq i \leq l \text{ and } \lambda_i = m\}. \tag{2.1.4}$$

The functions $C_m^{(n)}$ depend only on the cycle-type of σ and are therefore class functions on S_n. The functions $C_m^{(n)}$ are random variables since we have endowed S_n with the uniform measure. The $C_m^{(n)}$ are well studied objects, see for instance [5]. We have for example,

Lemma 2.1.8. *We have for $1 \le m \le n$*

$$\mathbb{E}\left[C_m^{(n)}\right] = \frac{1}{m} \tag{2.1.5}$$

and

Lemma 2.1.9. *Let $c_1, c_2, \cdots, c_n \in \mathbb{N}$ be given. Then*

$$\mathbb{P}\left[(C_1 = c_1, \cdots, C_n = c_n)\right] = \prod_{m=1}^{n} \left(\frac{1}{m}\right)^{c_m} \frac{1}{c_m!} \mathbf{1}\left(\sum_{m=1}^{n} m c_m = n\right). \tag{2.1.6}$$

The asymptotic behavior of $C_m^{(n)}$ is also well known

Lemma 2.1.10. *The random variables $C_m^{(n)}$ converge for each $m \in \mathbb{N}$ in distribution to a Poisson distributed random variable Y_m with $\mathbb{E}[Y_m] = \frac{1}{m}$. In fact, we have for all $b \in \mathbb{N}$*

$$(C_1^{(n)}, C_2^{(n)}, \cdots, C_b^{(n)}) \xrightarrow{d} (Y_1, Y_2, \cdots, Y_b) \qquad (n \to \infty),$$

with all Y_m independent.

2.1.3 The Feller-coupling

We need the Feller-coupling in the proof of Theorem 3.2.1 and in the probabilistic proof of theorem 5.8.1.
One of the problems in the definition of convergence in distribution of a sequence $(X_n)_{n\in\mathbb{N}}$ is that all X_n can be defined on a different probability spaces. Therefore it is very difficult to compare X_n with X_{n+1} directly. This is the case for $C_m^{(n)}$ and $C_m^{(n+1)}$.
Fortunately, the Feller-coupling constructs a probability space and new random variables $C_m^{(n)}$ and Y_m on this space, which have the same distributions as the $C_m^{(n)}$ and Y_m above and which can be easily compared. Many more details on the Feller-coupling can be found in [5].
The construction works as follows: Let $\xi := (\xi_1\xi_2\xi_3\xi_4\xi_5\cdots)$ be a sequence of independent Bernoulli-random variables with $\mathbb{P}[\xi_m = 1] = \frac{1}{m}$ and $\mathbb{P}[\xi_m = 0] = 1 - \frac{1}{m}$ An $m-$spacing is a sequence of $m-1$ consecutive zeroes in ξ or its truncations:

$$1 \ \underbrace{0\cdots0}_{m-1 \text{ times}} \ 1.$$

Definition 2.1.11. *Let $C_m^{(n)}(\xi)$ be the number of m-spacings in $1\xi_2\cdots\xi_n 1$. We define $Y_m(\xi)$ to be the number of m-spacings in the whole sequence ξ.*

Theorem 2.1.12. *The following holds:*

- *The above-constructed $C_m^{(n)}(\xi)$ have the same distribution as the $C_m^{(n)}(\lambda)$ in definition 2.1.7.*
- *$Y_m(\xi)$ is a.s. finite and Poisson distributed with $\mathbb{E}[Y_m(\xi)] = \frac{1}{m}$.*
- *All $Y_m(\xi)$ are independent.*
- $\mathbb{E}\left[\left|C_m^{(n)}(\xi) - Y_m(\xi)\right|\right] \le \frac{2}{n+1}$.
- *For any fixed $b \in \mathbb{N}$,*

$$\mathbb{P}\left[(C_1^{(n)}(\xi), \cdots, C_b^{(n)}(\xi)) \neq (Y_1(\xi), \cdots, Y_b(\xi))\right] \to 0 \ (n \to \infty).$$

Proof. We just illustrate the first point and refer the reader for a full proof to the book [5]. We consider for concreteness $n = 10$. Let us write down a permutation with cycles. We can start with "(1" and then make a ten-way choice between "(1)(2", "(1 2", "(1 3",...,"(1 10". We continue with a nine-way choice etc. We are only interested in the length of the cycles and not in the cycles itself. Therefore we just need to know when a cycle ends. We write a 1 for closing a cycle and 0 otherwise. We get a sequence of the form "100100010011". We reverse the order of the sequence since we prefer to read from left to right. Cycles of length m in $\sigma \in \mathcal{S}_n$ are mapped bijectively by this construction to $m-$spacings in the sequence $1\xi_2 \cdots \xi_n 1$. □

We write $C_m^{(n)}$ and Y_m for both sets of random variables and stress when we use the Feller-coupling.
One might guess from the definition that $C_m^{(n)} \leq Y_m$, but this is not true. It is indeed possible that $C_m^{(n)} = Y_m + 1$, but this can only happen if $\xi_{n-m} \cdots \xi_{n+1} = 10 \cdots 0$. If n is fixed, we have at most one such m with $C_m^{(n)} = Y_m + 1$. In order to state the following lemma, we set

$$B_m^{(n)} := \{\xi : \xi_{n-m} \cdots \xi_{n+1} = 10 \cdots 0\}\,. \tag{2.1.7}$$

Lemma 2.1.13. *We have:*

- $C_m^{(n)} \leq Y_m + \mathbf{1}_{B_m^{(n)}}$,
- $\mathbb{E}\left[\mathbf{1}_{B_m^{(n)}}\right] = \frac{1}{n+1}$,
- $C_m^{(n)}$ *does not converge a.s. to* Y_m.

Proof. The first point follows from the above considerations. The second point is a simple calculation using the independence of ξ_i. We illustrate the proof of the last point with an example. Let $\xi = (100010001 \cdots)$ be given. Then

n	1	2	3	4	5	6	7	8
$C_1^{(n)}$	1				1			
$C_2^{(n)}$		1				1		
$C_3^{(n)}$			1				1	
$C_4^{(n)}$				1	1	1	1	2

The general case is completely similar to this example. Let $\xi_v \xi_{v+1} \cdots \xi_{v+m+1} = 10 \cdots 01$. We then have for $1 \leq m_0 \leq m-1$ and $v \leq n \leq v + m + 1$

$$C_{m_0}^{(n)} = \begin{cases} C_{m_0}^{(v)} + 1, & \text{if } n = v + m_0, \\ C_{m_0}^{(v)}, & \text{if } n \neq v + m_0. \end{cases}$$

Since all $Y_m < \infty$ a.s. and $\sum_{m=1}^{\infty} Y_m = \infty$ a.s. we are done. □

2.1.4 Embedding of S_n into $U(n)$

We are interested in section 4 in the characteristic polynomial on S_n. Since the characteristic polynomial is defined as $\det(I - xg)$ we have to interpret the elements of S_n as matrices. We do this as follows: we define for $\sigma \in S_n$

$$g = g(\sigma) = (\delta_{i,\sigma(j)})_{1 \leq i,j \leq n} \tag{2.1.8}$$

with $\delta_{i,j} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$ It is easy to see that this map is an injective group homomorphism. We write for both groups S_n. We use this interpretation exclusively when we are looking at the characteristic polynomial $Z_n(x)$. The functions $W^n(f), W^{1,n}(f)$ and $W^{2,n}(f)$ in the sections 3 and 5 are independent of this interpretation.

2.2 Elementary analysis

We give here some simple results from analysis that we need in section 3. We state them without further comments.

Lemma 2.2.1 (Abel's partial summation)**.** *Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be complex numbers. Then*

$$\sum_{m=1}^{n} a_m b_m = A_n b_n + \sum_{m=1}^{n-1} A_m (b_{m+1} - b_m) \tag{2.2.1}$$

with $A_m := \sum_{k=1}^{m} a_k$ for $m \geq 1$ and $A_0 := 0$.

Proof. The idea of the proof is to write $a_m = A_{m+1} - A_m$ in the first sum and then split the sum. More details can be found in [14]. □

We can use this to prove

Lemma 2.2.2. *Let $(a_m)_{m=1}^{n}$ be a finite sequence and $A(s) := \sum_{m \leq s} a_m$. We then have*

$$\sum_{m=1}^{n} \frac{a_m}{m} = \left(\frac{1}{n} \sum_{m=1}^{n} a_m \right) + \int_1^n A(s) \frac{1}{s^2} \, ds. \tag{2.2.2}$$

Proof. Use lemma 2.2.1 with $b_m = \frac{1}{m}$. The rest is a straightforward verification. □

Lemma 2.2.3. *Let $(a_m)_{m=1}^{\infty}$ be a complex sequence. If $\frac{1}{n} \sum_{m=1}^{n} a_m \to L$ then*

$$\frac{1}{\log(n)} \sum_{m=1}^{n} \frac{a_m}{m} \to L. \tag{2.2.3}$$

Proof. This follows from lemma 2.2.2 with a small calculation. □

Finally we shall also need

Lemma 2.2.4 (Jensen's formula)**.** *Let f ba a holomorphic function in the disc $B_r(0) = \{x \in \mathbb{C}; |x| < r\}$ with $a_1, a_2, \cdots, a_l \in B_r(0)$ the zeros of f repeated according to multiplicity and $f(0) \neq 0$. Then*

$$\log|f(0)| = -\sum_{k=1}^{l} \log\left(\frac{r}{|a_k|} \right) + \int_{-1/2}^{1/2} \log|f(re^{2\pi i t})| \, dt. \tag{2.2.4}$$

Proof. See [2]. □

2.3 Uniformly distributed sequences

We use in section 3 some facts about uniform distributed sequences. We follow the book [20] and present the things we need in section 3. We omit or give only a short overview over the proofs that can be found in the book [20].

2.3.1 Definition and properties

We begin with the definition of uniformly distributed sequences.

Definition 2.3.1. *Let $\boldsymbol{t} = (t_m)_{m=1}^{\infty}$ be a sequence of real numbers in the compact interval $[a, b]$ (with $a < b$). For $a \leq \alpha \leq \beta \leq b$, define*

$$A([\alpha, \beta], n) = A([\alpha, \beta], n, \boldsymbol{t}) := \#\{1 \leq m \leq n; t_m \in [\alpha, \beta]\}. \tag{2.3.1}$$

The sequence $\boldsymbol{t} = (t_m)_{m\in\mathbb{N}}$ is called uniformly distributed in $[a, b]$ if we have

$$\lim_{n\to\infty} \left| \frac{A([\alpha, \beta], n)}{n} - \frac{(\beta - \alpha)}{b - a} \right| = 0 \tag{2.3.2}$$

for each α, β with $a \leq \alpha \leq \beta \leq b$.

It usual to work on the interval $[0, 1]$. We have introduced here this more general version since we have to restrict our sequences to subintervals (see section 2.3.2)

The following theorem shows that the name uniformly distributed is well chosen.

Theorem 2.3.2. *Let $h : [a, b] \to \mathbb{C}$ be a Riemann-integrable function and $\boldsymbol{t} = (t_m)_{m\in\mathbb{N}}$ be a uniformly distributed sequence in $[a, b]$. Then*

$$\frac{1}{n} \sum_{m=1}^{n} h(t_m) \to \frac{1}{b - a} \int_a^b h(s)\ ds. \tag{2.3.3}$$

Proof. The proof can be found in [20]. The Idea is to look at functions of the form $I_{[\alpha,\beta]}(s) := \begin{cases} 1 & \text{if } \alpha \leq s \leq \beta, \\ 0 & \text{otherwise.} \end{cases}$ with $a \leq \alpha < \beta \leq b$. The theorem is true for such functions since $\frac{1}{b-a} \int_a^b I_{[\alpha,\beta]}(s)\ ds = \frac{(\beta-\alpha)}{b-a}$. The theorem now follows by an approximation argument. □

We are primary interested in sequences of the form $(\{mt\})_{m=1}^{\infty}$ with

$$\{t\} := t - [t] \text{ and } [t] := \max\{n \in \mathbb{Z}, n \leq t\}. \tag{2.3.4}$$

The next lemma shows that the sequence $(\{mt\})_{m=1}^{\infty}$ is for almost all $t \in \mathbb{R}$ uniformly distributed.

Lemma 2.3.3. *Let $t \in \mathbb{R}$ be an irrational number. The sequence $(\{mt\})_{m\in\mathbb{N}}$ is uniformly distributed in $[0, 1]$.*

We also introduce the definition of discrepancy

Definition 2.3.4. *Let $(t_m)_{m=1}^{\infty}$ be a sequence of real numbers in the compact interval $[a, b]$. We define*

$$D_n = D_n([a, b], \boldsymbol{t}) := \sup_{a\leq\alpha\leq\beta\leq b} \left| \frac{A([\alpha, \beta], n)}{n} - \frac{(\beta - \alpha)}{b - a} \right|, \tag{2.3.5}$$

$$D_n^* = D_n^*([a, b], \boldsymbol{t}) := \sup_{a\leq\beta\leq b} \left| \frac{A([a, \beta], n)}{n} - \frac{(\beta - a)}{b - a} \right|. \tag{2.3.6}$$

We call D_n the discrepancy and D_n^* the $*$-discrepancy of the sequence $\boldsymbol{t}$.

It is easy to see that $D_n^* \leq D_n \leq 2D_n^*$ and therefore D_n and D_n^* are equivalent. We prefer to work with D_n^* since we have a more explicit expression for it.

Lemma 2.3.5. *Let n be fixed, $[a,b] = [0,1]$ and $\boldsymbol{t} = (t_m)_{m\in\mathbb{N}}$ be a sequence in the interval $[0,1]$. We define $0 \leq y_1 \leq \cdots \leq y_n \leq 1$ be the ascending ordered sequence $(t_m)_{m=1}^n$. We then have*

$$D_n^*([0,1], \boldsymbol{t}) = \max_{1\leq m\leq n} \max\left(\left|y_m - \frac{m}{n}\right|, \left|y_m - \frac{m-1}{n}\right|\right). \tag{2.3.7}$$

Proof. See [20]. □

The next lemma shows that the discrepancy is compatible with simple coordinate changes.

Lemma 2.3.6. *Let $\boldsymbol{t} = (t_m)_{m\in\mathbb{N}}$ be a sequence in the interval $[0,1]$ and $-\infty < a < b < \infty$. We define $y_m := (b-a)t_m + a$ and set $\boldsymbol{y} = (y_m)_{m\in\mathbb{N}}$. Then*

$$D_n([0,1], \boldsymbol{t}) = D_n([a,b], \boldsymbol{y}) \tag{2.3.8}$$

$$D_n^*([0,1], \boldsymbol{t}) = D_n^*([a,b], \boldsymbol{y}). \tag{2.3.9}$$

Proof. Let $0 \leq \alpha \leq \beta \leq 1$. It is obvious that

$$A([\alpha,\beta], n, \mathbf{t}) = A([a + (b-a)\alpha, a + (b-a)\beta], n, \mathbf{y}).$$

Therefore

$$\begin{aligned}
D_n([a,b], \mathbf{y}) &= \sup_{a\leq\gamma\leq\delta\leq b} \left|\frac{A([\gamma,\delta], n, \mathbf{y})}{n} - \frac{(\delta-\gamma)}{b-a}\right| \\
&= \sup_{0\leq\alpha\leq\beta\leq 1} \left|\frac{A([a+(b-a)\alpha, a+(b-a)\beta], n, \mathbf{y})}{n} - \frac{\big((a+(b-a)\beta) - (a+(b-a)\alpha)\big)}{b-a}\right| \\
&= \sup_{0\leq\alpha\leq\beta\leq 1} \left|\frac{A([\alpha,\beta], n, \mathbf{t})}{n} - \frac{(\beta-\alpha)}{1}\right| = D_n([0,1], \mathbf{t}).
\end{aligned}$$

The proof for D_n^* is similar. □

An important thing is that Theorem 2.3.2, the discrepancy and uniform distributed sequences are closely related. In fact we have

Lemma 2.3.7. *Let $\boldsymbol{t} = (t_m)_{m=1}^\infty$ be a sequence in the compact interval $[a,b]$. Then the following are equivalent*

1. *$\boldsymbol{t}$ is uniformly distributed in $[a,b]$,*
2. *$\lim_{n\to\infty} D_n([a,b], \boldsymbol{t}) = 0$.*
3. *Let $h : [a,b] \to \mathbb{C}$ be an arbitrary Riemann-integrable function. Then*

$$\frac{1}{n}\sum_{m=1}^n h(t_m) \to \frac{1}{b-a}\int_a^b h(s)\ ds \quad \text{for } n \to \infty$$

We have introduced the discrepancy since it allows us to estimate the rate of convergence in theorem 2.3.2. We have

Theorem 2.3.8 (Koksma's inequality)**.** *Let $h : [a,b] \to \mathbb{C}$ be a real analytic function with $V(h) = \int_a^b \left|\frac{d}{ds}h(s)\right| \, ds$. Let $\boldsymbol{t} = (t_m)_{m\in\mathbb{N}}$ be a arbitrary sequence in $[a,b]$. Then*

$$\left| \frac{1}{n} \sum_{m=1}^{n} h(t_m) - \frac{1}{b-a} \int_a^b h(s) \, ds \right| \le V(h) D_n^*([a,b], \boldsymbol{t}). \tag{2.3.10}$$

In this thesis, we shall work with functions of the form $h(s) = \log\bigl(r(s)\bigr)$ with $r(s)$ a real analytic function. A simple calculation shows that $\log\bigl(r(s)\bigr) \sim C_1 \log(s - s_0)$ and $\frac{d}{ds}\log\bigl(r(s)\bigr) \sim C_2 \frac{1}{(s-s_0)}$ for $s \to s_0$ and s_0 a zero of $r(s)$. We therefore cannot use Koksma's inequality in this situation. We instead use

Theorem 2.3.9. *Let $h :]a,b[\to \mathbb{C}$ be a real analytic function such that*

$$\int_{a+\delta}^{b-\delta} |h(s)| \, ds < \infty \text{ and } \int_{a+\delta}^{b-\delta} \left|\frac{d}{ds}h(s)\right| \, ds < \infty \text{ for all } 0 < \delta < \frac{b-a}{2} \tag{2.3.11}$$

and $\boldsymbol{t} = (t_m)_{m\in\mathbb{N}}$ be a sequence in $]a,b[$.
Let $n \in \mathbb{N}$ be arbitrary and $\delta > 0$ such that

$$a + \delta < \min_{1\le m\le n} t_m < \max_{1\le m\le n} t_m < b - \delta.$$

We then have

$$\left| \frac{1}{n} \sum_{m=1}^{n} h(t_m) - \frac{1}{b-a} \int_{a+\delta}^{b-\delta} h(s) \, ds \right| \le \delta |h(a+\delta)| + \delta |h(b-\delta)|$$
$$+ D_n^*([a,b], \boldsymbol{t}) \int_{a+\delta}^{b-\delta} \left| \frac{d}{ds} h(s) \right| \, ds. \tag{2.3.12}$$

Proof. This theorem was already proven in [17] and we therefore give only a short overview. W.l.o.g. let $[a,b] = [0,1]$. We define $y_1 \le \cdots \le y_n$ be the ascending ordered sequence $(t_m)_{m=1}^n$. We then know from lemma 2.3.5 that

$$D_n^*([0,1], \mathbf{t}) = \max_{1\le m\le n} \max \left(\left| y_m - \frac{m}{n} \right|, \left| y_m - \frac{m-1}{n} \right| \right).$$

We put $y_0 := \delta, y_{n+1} := 1 - \delta$ and look at

$$\sum_{m=0}^{n} \int_{y_m}^{y_{m+1}} (s - \frac{m}{n}) \frac{d}{ds} h(s) \, ds. \tag{2.3.13}$$

The theorem now follows by integration by parts. □

2.3.2 Restriction of sequences

Most of the functions we consider in this thesis have the form $\log\bigl(f(e^{2\pi i s})\bigr)$. The most problematic points are of course the zeros of $f(e^{2\pi i s})$. Theorem 2.3.9 allows us to handle the case when the only zeros are $s = 0$ and $s = 1$. The idea to solve more general cases is to split the interval into several pieces such that the zeros of $f(e^{2\pi i s})$ are the boundary points and then apply theorem 2.3.9 separately to each piece. To do this, we have to define

Definition 2.3.10. *Let* $\boldsymbol{t} = (t_m)_{m\in\mathbb{N}}$ *be a sequence in the interval* $[a,b]$ *and* $[c,d] \subset [a,b]$. *We put*

$$\begin{aligned} y_1 &:= t_{m_1} \text{ with } m_1 := \min\{m \in \mathbb{N}; t_m \in [c,d]\}, \\ y_2 &:= t_{m_2} \text{ with } m_2 := \min\{m > m_1; t_m \in [c,d]\}, \\ y_3 &:= t_{m_3} \text{ with } m_3 := \min\{m > m_2; t_m \in [c,d]\}, \\ &\vdots \\ s &:= \sup\{k \in \mathbb{N}; y_k \text{ exists.}\} \end{aligned}$$

and define $\boldsymbol{y} := (y_m)_{m=1}^{s}$. *We call* $\boldsymbol{y}$ *the restricted sequence (to* $[c,d]$*) and write* $\boldsymbol{t} \cap [c,d]$ *for it.*

It is possible that $\mathbf{y}$ is finite or empty, but we will always have that $\mathbf{y}$ is infinite. The next lemma shows that the discrepancy and uniform distributed sequences behaves well under restriction.

Lemma 2.3.11. *Let* $\boldsymbol{t} = (t_m)_{m\in\mathbb{N}}$ *be a sequence in* $[a,b]$ *and* $[c,d] \subset [a,b]$ *with* $c < d$.

1. *If* $\boldsymbol{t}$ *is uniformly distributed in* $[a,b]$ *then* $\boldsymbol{y} = \boldsymbol{t} \cap [c,d]$ *is uniformly distributed in* $[c,d]$.
2. *If* $D_n([a,b],\boldsymbol{t}) = O(n^{-\alpha})$ *for some* $\alpha > 0$ *then we have also* $D_n([c,d],\boldsymbol{y}) = O(n^{-\alpha})$.

Proof. It is easy to see that we can reformulate the definition of m_k in definition 2.3.10 as follows

$$m_k = \min\{n \in \mathbb{N}; A([c,d],n,\mathbf{t}) = k\}. \tag{2.3.14}$$

Since the sequence $\mathbf{t}$ is uniformly distributed in $[a,b]$ we have

$$\lim_{n\to\infty} \frac{A([c,d],n,\mathbf{t})}{n} = \frac{d-c}{b-a} \tag{2.3.15}$$

and therefore

$$A([c,d],n,\mathbf{t}) \sim n\frac{d-c}{b-a}. \tag{2.3.16}$$

It follows that the sequence $\mathbf{y}$ is infinite and there exists a sequence $(k_n)_{n\in\mathbb{N}}$ with $n = A([c,d],k_n,\mathbf{t})$. We then have for all α,β with $c \le \alpha \le \beta \le d$

$$A([\alpha,\beta],n,\mathbf{y}) = A([\alpha,\beta],k_n,\mathbf{t}). \tag{2.3.17}$$

It follows from (2.3.16) that

$$n = A([c,d],k_n,\mathbf{t}) \sim k_n\frac{d-c}{b-a}. \tag{2.3.18}$$

This shows that $k_n = O(n)$ and we get

$$\lim_{n\to\infty} \frac{A([\alpha,\beta],n,\mathbf{y})}{n} = \lim_{n\to\infty} \frac{A([\alpha,\beta],k_n,\mathbf{t})}{k_n}\frac{k_n}{n} = 0.$$

The last equality holds since $\mathbf{t}$ is uniformly distributed in $[a,b]$ and $\frac{k_n}{n}$ is convergent. This proves 1. The proof of 2 is similar. One has to replace (2.3.15) by

$$\frac{A([c,d],n,\mathbf{t})}{n} - \frac{d-c}{b-a} = O(n^{-\alpha}) \tag{2.3.19}$$

This is true by the assumption on $D_n([a,b],\mathbf{t})$. We therefore have

$$\frac{n}{k_n} = \frac{A([c,d],k_n,\mathbf{t})}{k_n} = \frac{d-c}{b-a} + O(k_n^{-\alpha}) = \frac{d-c}{b-a} + O(n^{-\alpha}). \tag{2.3.20}$$

We get

$$\begin{aligned}
\left|\frac{A([\alpha,\beta],n,\mathbf{y})}{n} - \frac{\beta-\alpha}{d-c}\right| &= \left|\frac{A([\alpha,\beta],k_n,\mathbf{t})}{k_n}\frac{k_n}{n} - \frac{\beta-\alpha}{d-c}\right| \\
&\le \left|\frac{A([\alpha,\beta],k_n,\mathbf{t})}{k_n}\frac{b-a}{d-c} - \frac{\beta-\alpha}{d-c}\right| + \left|\frac{A([\alpha,\beta],k_n,\mathbf{t})}{k_n}\, O(n^{-\alpha})\right| \\
&= \frac{b-a}{d-c}\left|\frac{A([\alpha,\beta],k_n,\mathbf{t})}{k_n} - \frac{\beta-\alpha}{b-a}\right| + O(n^{-\alpha}) \\
&\le D_{k_n}([a,b],\mathbf{t}) + O(n^{-\alpha}) = O(n^{-\alpha}).
\end{aligned} \tag{2.3.21}$$

This proves 2 since all $O(.)$ appearing (2.3.21) are independent of α and β. □

2.4 Diophantine approximation

As mentioned before, the sequence $(\{mt\})_{m\in\mathbb{N}}$ is the most important sequence we look at. Since our target is to use theorem 2.3.9, we have to choose a $\delta = \delta(n)$ with $\delta \le \{mt\} \le 1-\delta$ for $1 \le m \le n$ and

$$D_n^*([a,b],\mathbf{t}) \int_{a+\delta}^{b-\delta} \left|\frac{d}{ds}h(s)\right| \, ds \longrightarrow 0 \ (n \to \infty)$$

To reach this, we use here some classical results from diophantine approximation. We put

Definition 2.4.1. *Let $t \in \mathbb{R}$ be arbitrary. We put $||t|| := \inf_{n\in\mathbb{Z}} |t-n|$ and*

$$\eta = \sup\left\{\gamma \in \mathbb{R}_+ : \liminf_{n\in\mathbb{N}} n^{\gamma}||nt|| = 0\right\}.. \tag{2.4.1}$$

The constant η is called the type of t. If η is finite then t is called of finite type. We call $x = e^{2\pi i t}$ of finite type, if t is of finite type.

Lemma 2.4.2. *Let t be irrational of type η. We then have*

1. *For each $k < \eta$ and $C > 0$ there exists infinitely many rational numbers $\frac{p}{q}$ with*

$$|t - \frac{p}{q}| \le \frac{C}{q^{k+1}}$$

2. *For each $k > \eta$ there exists a constant $C > 0$ such that for each $m \in \mathbb{N}$*

$$||mt|| > \frac{C}{m^k}$$

3. *For each $k > \eta, q \in \mathbb{N}$ there exists a constant $C > 0$ such that for each $m, p \in \mathbb{N}$*

$$|mt - \frac{p}{q}| > \frac{C}{m^k}$$

Proof. This follows directly from the definition 2.4.1 and a small calculation. □

We can also estimate the discrepancy of $(\{mt\})_{m\in\mathbb{N}}$ if t is of finite type.

Theorem 2.4.3. *Let x be of finite type η and $\boldsymbol{t} = (\{mx\})_{m\in\mathbb{N}}$. We then have for each $\epsilon > 0$*

$$D_n([0,1], \boldsymbol{t}) = O\left(n^{-\frac{1}{\eta}+\epsilon}\right). \tag{2.4.2}$$

Proof. See [20]. □

2.5 Probability

We use here only some well known definitions and facts from probability theory that can be found in [6] or in [16]. We therefore omit the proofs. We introduce here very short the definition of random variables and weak convergence. We also give two tools to decide if a sequence $(X_n)_{n\in\mathbb{N}}$ converge weakly to some X.

Definition 2.5.1. *Let Ω be a set and $\mathfrak{A} \subset 2^{\Omega}$ be a $\sigma-$algebra. A function $\mathbb{P} : \mathfrak{A} \to \mathbb{R}_{\geq 0}$ is called a probability measure if*

1. $\mathbb{P}[\Omega] = 1$,
2. $\mathbb{P}[\varnothing] = 0$,
3. *If $(A_i)_{n\in\mathbb{N}}$ is sequence in $\mathfrak{A}$ with all A_i pairwise disjoint. then*

$$\mathbb{P}\left[\bigcup_{i=1}^{\infty} A_i\right] = \sum_{i=1}^{\infty} \mathbb{P}[A_i].$$

We call $(\Omega, \mathfrak{A}, \mathbb{P})$ a probability space.

Definition 2.5.2. *Let $(\Omega, \mathfrak{A}, \mathbb{P})$ be a probability space and T be a topological. A function $X : \Omega \to T$ is called a random variable if*

$$X^{-1}(U) \in \mathfrak{A} \text{ for all } U \text{ open.} \tag{2.5.1}$$

We use here only the cases $T = \mathbb{R}$ and $T = \mathbb{C}$.

Definition 2.5.3. *Let X be a random variable on the probability space $(\Omega, \mathfrak{A}, \mathbb{P})$. The cumulative distribution functions F of a real valued random variable is defined as*

$$F(x) := \mathbb{P}[X \leq x] \text{ for all } x \in \mathbb{R}. \tag{2.5.2}$$

The cumulative distribution function a complex valued random variable is defined as.

$$F(x,y) = \mathbb{P}[\mathrm{Re}(X) \leq x, \mathrm{Im}(X) \leq y] \tag{2.5.3}$$

We call X an (absolutely) continuous if there exists a function f with

$$F(x) = \int_{-\infty}^{x} f(x)\, dx \text{ for all } x \in \mathbb{R}, \tag{2.5.4}$$

resp.

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y)\, dxdy \text{ for all } x, y \in \mathbb{R}. \tag{2.5.5}$$

The function f is called the density of X.

Definition 2.5.4. *A real valued random variable X is called Gaussian or said to be normal distributed if it has the density*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} \tag{2.5.6}$$

If $\mu = 0, \sigma = 1$ then X is called a standard Gaussian random variable.

2.5.1 Simple lemmas

We now give two simple lemmas we need in section 5.

Lemma 2.5.5 (Borel-Cantelli)**.** *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of events. If $\sum_{n\in\mathbb{N}} \mathbb{P}[A_n] < \infty$ then*

$$0 = \mathbb{P}[\limsup(A_n)] \tag{2.5.7}$$

with $\limsup(A_n) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$.

Lemma 2.5.6 (Markov's inequality)**.** *Let X be a real valued random variable and $x \in \mathbb{R}_+$. Then*

$$\mathbb{P}[|X| > x] \leq \frac{\mathbb{E}[|X|]}{x}. \tag{2.5.8}$$

2.5.2 Weak convergence of Random variables

The convergence of random variables is an important concept in probability theory. There exists several different notions of convergence of sequences of random variables like point-wise convergence, convergence in probability and weak convergence. We need here only the weak convergence of random variables.

Definition 2.5.7. *Let $(X_n)_{n\in\mathbb{N}}$ and X be random variables, not necessarily defined on a common probability space. The sequence $(X_n)_{n\in\mathbb{N}}$ is said to converge weak to X (written $X_n \xrightarrow{d} X$) if*

$$\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)] \tag{2.5.9}$$

for all all f continuous and bounded.

For real random variables, definition 2.5.7 is equivalent to

$$\lim_{n\to\infty} F_n(x) = F(x), \tag{2.5.10}$$

where F_n resp. F for the cumulative distribution function of X_n resp. of X. It is often difficult to check (2.5.9) directly or to write down a explicit expression for the cumulative distributions. It is therefore not so easy to decide directly if $X_n \xrightarrow{d} X$. We now give two tools to decide if $X_n \xrightarrow{d} X$.

2.5.2.1 Characteristic function and Lévy's continuity theorem

We set

Definition 2.5.8. *Let X be a real valued random variable. The characteristic function of X is defined as*

$$\varphi(t) = \mathbb{E}\left[e^{itX}\right] \quad \forall t \in \mathbb{R}. \tag{2.5.11}$$

An important example of a characteristic function is

Lemma 2.5.9. *Let X be a normal distributed random variable. Then the characteristic function φ of X is given by*

$$\varphi(t) = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}. \tag{2.5.12}$$

We know list some important properties of characteristic functions.

Lemma 2.5.10. *Let X be a real valued random variable with characteristic function φ. Then*

1. *The characteristic function of a real valued random variable always exists (since $|e^{itX}| = 1$),*
2. *φ is a continuous function in t on $\mathbb{R}$,*
3. *$\varphi(0) = 1$,*
4. *$|\varphi(t)| \leq 1$.*

The characteristic function is also useful to decide if two random variables are independent. We have

Lemma 2.5.11. *Let X_1, X_2 be random variables with characteristic functions φ_1, φ_2. Then the following are equivalent*

- *X_1 and X_2 are independent,*
- *$\mathbb{E}\left[e^{it_1X_1+it_2X_2}\right] = \varphi_1(t_1)\varphi_2(t_2)$.*

The definition of the characteristic function cannot be directly extended to complex valued random variables since $\mathbb{E}\left[e^{itX}\right]$ can be undefined. There exists several ways to define the characteristic function in the complex case. We use here

Definition 2.5.12. *Let $X = X_1 + iX_2$ be a complex valued random variable. The characteristic function of X is defined as*

$$\varphi(t_1, t_2) = \mathbb{E}\left[e^{it_1X_1+it_2X_2}\right] \quad \forall t_1, t_2 \in \mathbb{R}. \tag{2.5.13}$$

The next theorem is one of the most important in probability theory. We state it just for complex random variables since this includes also the real case.

Theorem 2.5.13 (Lévy's continuity theorem.)**.** *Let a sequence of complex valued random variables $(X_n)_{n=1}^{\infty}$ be given, not necessarily defined on a common probability space. We write φ_n for the characteristic function of X_n. If the sequence of characteristic functions converges point-wise to some function φ*

$$\varphi_n(t_1, t_2) \to \varphi(t_1, t_2) \quad \forall t_1, t_2 \in \mathbb{R}. \tag{2.5.14}$$

Then it is equivalent:

- *X_n converges weakly to some random variable X.*

- *φ is a characteristic function of some random variable X.*
- *$\varphi(t_1, t_2)$ is a continuous function of t_1, t_2.*
- *$\varphi(t_1, t_2)$ is continuous at $t_1, t_2 = 0$.*

Proof. See [6] or [16]. □

This theorem is named after a French probabilist Paul Pierre Lévy (1886 − 1971). Lévy's continuity theorem shows that the convergence of $\varphi_n(t)$ implies $X_n \xrightarrow{d} X$ and the non convergence of $\varphi_n(t)$ implies that $X_n \overset{d}{\nrightarrow} X$.

2.5.2.2 Method of moments

It is not always possible to calculate the characteristic function explicitly. In this case one can use the method of moments. We set

Definition 2.5.14. *Let X be a real valued random variable such that all moments*

$$s_k = \mathbb{E}\left[X^k\right] \tag{2.5.15}$$

are finite. We say that X is uniquely determined by its moments if

$$\mathbb{E}\left[Y^k\right] = s_k \quad \forall k \in \mathbb{N} \implies X \overset{d}{=} Y. \tag{2.5.16}$$

One possibility to decide if X be can uniquely determined by its moments is to use Carleman's condition, named after Torsten Carleman (1892 – 1949).

Theorem 2.5.15 (Carleman's condition)**.** *Let X be a real valued random variables with finite moments s_k. If*

$$\sum_{k=1}^{\infty} s_{2k}^{-\frac{1}{2k}} = +\infty, \tag{2.5.17}$$

then X is uniquely determined by its moments.

Proof. See [16]. □

We now state

Theorem 2.5.16. *Let $(X_n)_{n=1}^{\infty}$, X be a real random variables. Suppose that*

$$\mathbb{E}\left[|X_n|^k\right] < \infty \text{ and } \mathbb{E}\left[|X|^k\right] < \infty \text{ for all } n, k \in \mathbb{N} \tag{2.5.18}$$

and that X is uniquely defined by its moments. If

$$\lim_{n\to\infty} \mathbb{E}\left[X_n^k\right] = \mathbb{E}\left[X^k\right] \text{ for all } k \in \mathbb{N} \tag{2.5.19}$$

then

$$X_n \xrightarrow{d} X. \tag{2.5.20}$$

This theorem is not so strong as theorem 2.5.13 since we need that X is uniquely defined by its moments. Also the non convergence of the moments does not imply $X_n \overset{d}{\nrightarrow} X$. A particular example is

Example 2.5.17. Let

$$f_n(x) := \frac{1}{1+\frac{1}{n}} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} + \mathbb{I}\Big(\big[n!, n! + \frac{1}{n} \big] \Big)(x) \right) \tag{2.5.21}$$

and X_n be a random variable with density f_n. It follows immediately from (2.5.10) that $X_n \xrightarrow{d} X$ where X is a standard Gaussian. It is also obvious that $\mathbb{E}[X_n] \to \infty$.

-

2.6 Generating functions

2.6.1 Definition of generating functions

The idea of generating functions is to encode information of a sequence into a formal power series. There exists several different type of generating functions.

Definition 2.6.1. *Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers.*

- *The ordinary generating function of $(h_n)_{n\in\mathbb{N}}$ is the formal power series*

$$h(t) = h(h_n, t) = \sum_{n=0}^{\infty} h_n t^n. \tag{2.6.1}$$

- *The exponential generating function of $(h_n)_{n\in\mathbb{N}}$ is the formal power series*

$$Eh(t) = Eh(h_n, t) = \sum_{n=0}^{\infty} h_n \frac{t^n}{n!}. \tag{2.6.2}$$

- *The poisson generating function of $(h_n)_{n\in\mathbb{N}}$ is the formal power series*

$$Ph(t) = Ph(h_n, t) = \sum_{n=0}^{\infty} h_n e^{-t} \frac{t^n}{n!}. \tag{2.6.3}$$

We use here only ordinary generating function and therefore call it just generating function. One can of course define generating functions in more than one variable, but we need only the one variable case here.

Definition 2.6.2. *If a formal power series $h(t) = \sum_{n=0}^{\infty} h_n t^n$ is given. We then define $[h]_n := h_n$.*
We use $[.]_n$ only with respect to t, also when h_n depends on other variables.

Example 2.6.3. Simple examples of generating functions are

1. We have for $h_n = 1$

$$h(t) = \sum_{n=0}^{\infty} t^n = \frac{1}{1-t} \tag{2.6.4}$$

2. We have for $h_n = (-1)^n$

$$h(t) = \sum_{n=0}^{\infty} (-1)^n t^n = \frac{1}{1+t} \tag{2.6.5}$$

3. Let $p \in \mathbb{N}$ be arbitrary and $h_n = \begin{cases} 1 & \text{if } p|n, \\ 0 & \text{otherwise.} \end{cases}$ Then

$$h(t) = \frac{1}{1 - t^p} \tag{2.6.6}$$

The reason why generating functions are useful is that it is often possible to write down the generating function $h(t)$ without knowing h_n explicitly. We do this in section 4.4 for $h_n = \mathbb{E}\left[Z_n^s(x)\right]$ and in section 5 for $h_n = \mathbb{E}\left[W^{1,n}(f)(x)\right]$ and for $h_n = \mathbb{E}\left[W^{2,n}(f)(x)\right]$.
A good and classical example to illustrate this is

Example 2.6.4. We denote the number of partitions of n by $P(n)$. The definition of $P(n)$ is very simple, but there is no combinatorial way to write down $P(n)$ explicitly for arbitrary $n \in \mathbb{N}$. But we can write down the generating function for $P(n)$. We have

$$\sum_{n=0}^{\infty} P(n) t^n = \prod_{k=1}^{\infty} \frac{1}{(1 - t^k)} \tag{2.6.7}$$

and both sides are convergent for $|t| < 1$.

Proof. The right side of 2.6.7 can be written as

$$\big(1 + t + t^2 + t^3 + \cdots\big)\big(1 + (t^2) + (t^2)^2 + (t^2)^3 + \cdots\big)\big(1 + (t^3) + (t^3)^2 + (t^3)^3 + \cdots\big) \cdots .$$

It is now easy to see that the coefficient of t^n is $P(n)$. □

2.6.2 Extraction of coefficients

If a generating function $h(t)$ is given then a natural question is of course: what is $[h]_n$ and what is the asymptotic behavior of $[h]_n$. If $h(t)$ is holomorphic near 0 then one can use Cauchy's integral formula to do this. Explicitly we have

$$[h]_n = \frac{1}{2\pi i} \oint_\gamma \frac{h(z)}{z^{n+1}}\, dz. \tag{2.6.8}$$

with γ a simple closed curve with winding number 1 about 0. A proof of this equation and further details can be found in the book *Complex analysis* [14].
Unfortunately it is often difficult to calculate the integral explicit or to get direct the asymptotic behavior for $n \to \infty$. This is can be seen by choosing $\gamma(s) = e^{2\pi i t}$. We then have $\frac{1}{z^n} = e^{-n 2\pi i t} = \cos(2\pi n t) + i \sin(2\pi n t)$ and there are many sign changes in the integrand.
It is possible to use (2.6.8) to get the asymptotic behavior. We now state a very general theorem in [13] basing on (2.6.8).

Definition 2.6.5. *Let* $1 < R$ *and* $0 < \varphi < \frac{\pi}{2}$ *be given. We then define*

$$\Delta_0 = \Delta_0(R, \varphi) = \{z \in \mathbb{C}; |z| < R, z \neq 1, |\arg(z-1)| > \varphi\} \tag{2.6.9}$$

Theorem 2.6.6. *Let* $\xi_1, \cdots, \xi_k \in \mathbb{C}$ *be given with* $|\xi_1| = |\xi_2| = \cdots = |\xi_k| = r > 0$. *Let* $R > r$, $0 < \varphi < \frac{\pi}{2}$ *an set*

$$D := D(r, R, \varphi) := \bigcap_{j=1}^{k} \xi_j \Delta_0(R/r, \varphi) \tag{2.6.10}$$

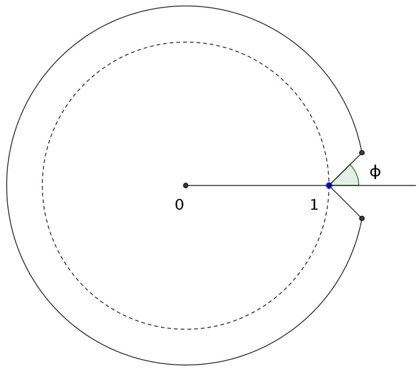

Figure 2.1: Illustration of Δ_0

Suppose that $h(t)$ is holomorphic in D and that there exist $\alpha_1, \cdots, \alpha_k \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ with

$$(t-\xi_j)^{\alpha_j} h(t) = c_j + O(t-\xi_j) \text{ for } t \to \xi_j, t \in D, c_j \neq 0 \tag{2.6.11}$$

for $1 \leq j \leq k$. We then have

$$[h]_n \sim \sum_{j=1}^{k} \xi_j^{-n} \frac{n^{\alpha_j - 1}}{\Gamma(\alpha_j)} \tag{2.6.12}$$

with $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

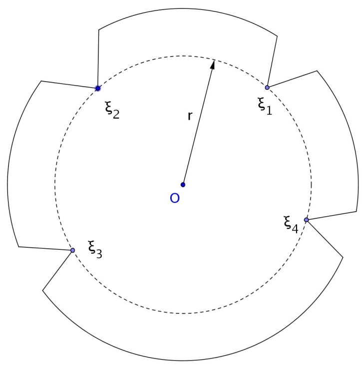

Figure 2.2: Illustration of D with four points $\xi_1, \cdots, \xi_4$

Proof. See [13], section 6. □

Theorem 2.6.6 is enough for our purposes, but we cannot apply it to the generating function $h(t) = \prod_{k=1}^{\infty} \frac{1}{(1-t^k)}$ since $h(t)$ has "infinitely many zeros on S^1". It is not possible to adapt the proof of theorem 2.6.6 to $\prod_{k=1}^{\infty} \frac{1}{(1-t^k)}$ since the main idea is to leave the unit disc. Nevertheless, Hardy and Ramanujan were able using this generating function to calculate the asymptotic behavior of $P(n)$.

Theorem 2.6.7. *(Hardy and Ramanujan,1917)*

$$P(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \tag{2.6.13}$$

Proof. See [4]. □

2.6.3 Lemmas to write down generating functions

We use here two tools to write down generating function.

First tool:

Lemma 2.6.8. *Let* $(a_m)_{m\in\mathbb{N}}$ *be a sequence of complex numbers. Then*

$$\sum_{\lambda} \frac{1}{z_\lambda} a_\lambda t^{|\lambda|} = \exp\left(\sum_{m=1} \frac{1}{m} a_m t^m\right) \text{ with } a_\lambda := \prod_{i=1}^{l(\lambda)} a_{\lambda_i}. \tag{2.6.14}$$

where $\sum_\lambda$ *denotes the sum over all partitions and*

$$z_\lambda := \prod_{r=1}^{n} r^{c_r} c_r! \text{ and } c_r = c_r(\lambda) := \#\{i|\lambda_i = r\}. \tag{2.6.15}$$

If RHS or LHS of (2.6.14) *is absolutely convergent then so is the other.*

Proof. The proof can be found in [22] or can be directly verified using the definitions of z_λ and the exponential function. □

Second tool

Let $\mathcal{N}$ be a set and $\mathbf{x} = (x_1, \cdots, x_s)$ be complex variables. We then define

$$f_{\mathcal{N}}(\mathbf{x}, t) = f_{\mathcal{N},wt}(\mathbf{x}, t) = \sum_{q\in\mathcal{N}} wt(q) \tag{2.6.16}$$

for a function $wt : \mathcal{N} \to \mathbb{C}[[\mathbf{x}, t]]$. The next theorem shows that $f_{\mathcal{N}}$ behaves well for disjoint sets.

Theorem 2.6.9. *Let* $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$ *be a set and* $wt : Q_1 \times Q_2 \to \mathbb{C}[[\mathbf{x}]]$ *such as* $wt((q_1, q_2)) = wt(q_1)wt(q_2)$, *then*

$$f_{Q_1\times Q_2}(\mathbf{x}) = f_{Q_1}(\mathbf{x}) f_{Q_2}(\mathbf{x}). \tag{2.6.17}$$

Proof. See [24] or verify it directly. □

2.7 Representation theory

We only need some very basic facts and definitions from representation theory. We present here a short introduction and illustrate the notation with examples. We use these examples in section 4. We omit the proofs since they can be found in most books about representation theory, for instance in the book *Lie Groups* by Bump [9].

Definition 2.7.1. *Let G be a finite group and let W be a complex vector space. A representation of G is a map $\pi : G \times W \to W$ with*

- $\pi_g(.)$ *is linear* $\forall g \in G$,
- $\pi_e(w) = w \ \forall w \in W$,
- $\pi_{gh}(w) = \pi_g(\pi_h(w)) \ \forall g, h \in G$ *and* $\forall w \in W$.

To simplify the notation, we write $gw := \pi_g(w)$.

We write W for a representation $\pi : G \times W \to W$ if the operation is clear.

Example 2.7.2. We first set $V = V^{(n)} := \mathbb{C}^n$ and label the canonical basis of V by $\{v_i\}$. We keep most of the time n fixed and write therefore V. We use the notation $V^{(n)}$ only when there is a danger of confusion with the dimension.
We interpret S_n as subgroup of the unitary group (see (2.1.8)) and write the vectors $v \in V$ as column vectors. Since $g \in S_n$ is a $n \times n$ matrix, we can use the usual matrix multiplication to define gv. It is obvious that this operation defines a representation of S_n. This representation is called the defining representation.
The action of $g = (\delta_{i,\sigma(j)})_{1 \leq i,j \leq n} \in S_n$ on the canonical basis v_i is given by

$$gv_i = v_{\sigma(i)}. \tag{2.7.1}$$

Remark: We use the notation $g \in S_n$ for the operation on a vector v and $\sigma \in S_n$ for the operation on a set $I \subset \{1, \cdots, n\}$.

Example 2.7.3. For $k \in \mathbb{N}$, we set $V_k := V_k^{(n)} := \bigwedge^k V$ and define

$$g(v_{i_1} \wedge \cdots \wedge v_{i_k}) := (gv_{i_1}) \wedge \cdots \wedge (gv_{i_k}) \quad \forall g \in S_n \tag{2.7.2}$$

For completeness, we set $V_0 := \mathbb{C}$ with $g1 = 1$ for all $g \in S_n$.
It is clear that this operation defines a representation $S_n \times V_k \to V_k$.
We define $\mathrm{SeT}_k = \mathrm{SeT}_k^{(n)} := \{I \subset \{1, \cdots, n\}; |I| = k\}$.
We put for $I \in \mathrm{SeT}_k$

$$v_I := v_{i_1} \wedge \cdots \wedge v_{i_k} \text{ with } i_j \in I \text{ and } i_1 < i_2 < \cdots < i_k \tag{2.7.3}$$

We know from linear algebra that the collection $\{v_I\}_{I \in \mathrm{SeT}_k}$ forms a basis of V_k. For $g = (\delta_{i,\sigma(j)})_{1 \leq i,j \leq n} \in S_n$ and $I \in \mathrm{SeT}_k$, we have

$$gv_I = v_{\sigma(I)} \text{ or } gv_I = -v_{\sigma(I)}. \tag{2.7.4}$$

One might guess $g(v_I) = \epsilon(\sigma)v_{\sigma(I)}$, but that is wrong. For $\sigma = (2, 4)$ we get

$$g(v_1 \wedge v_2) = v_1 \wedge v_4 \neq \epsilon(\sigma)v_1 \wedge v_4,$$

$$g(v_2 \wedge v_3) = v_4 \wedge v_3 = \epsilon(\sigma)v_3 \wedge v_4.$$

We will need the sign exclusively in the case of $\sigma(I) = I$. Then indeed we have

$$g(v_I) = \epsilon(\sigma)v_I.$$

We will extend the definitions of V_k and SeT_k to arbitrary $\mu \in \mathbb{N}^s$ (see definitions 4.1.1 and 4.3.2).
Remark: We write V_k instead of $\bigwedge^k V$ since it is shorter and corresponding indices are always on the same place.

Definition 2.7.4. *Let W be a representation. A subspace $Z \subset W$ is called $G-$invariant (or just invariant), if*

$$gz \in Z \quad \forall z \in Z, \forall g \in G.$$

Example 2.7.5. Each representation W has at least two invariant subspaces, namely W itself and 0. They are called the trivial ones.

Definition 2.7.6. *A representation W is called irreducible, if 0 and W are the only $G-$invariant subspace of W. Otherwise the representation is called reducible.*

Example 2.7.7. We continue with example 2.7.2. We define $d := \sum_{i=1}^{n} v_i$. Since each $g \in S_n$ permutes the basis v_i, we have $gd = d$. Therefore $D := \mathbb{C}d$ is an $G-$invariant subspace of V and so V is reducible for $n > 1$.

Definition 2.7.8. *Let W_1 and W_2 be representations of G. We say that W_1 is* isomorphic *to W_2 (written $W_1 \cong W_2$), if there is a vector space isomorphism $\varphi : W_1 \to W_2$ with $\varphi(gw) = g\varphi(w)$ for all $g \in G$.*

Lemma 2.7.9. *Each complex representation is isomorphic to a direct sum of irreducible representations.*

Example 2.7.10. We continue with example 2.7.2. D is clearly isomorphic to $\mathbb{C}$.
We set $B := V/D$. We define $g(v + D) := gv + D$ for each $v + D \in B, g \in S_n$. It is easy to see that this operation is well defined and that B becomes a representation of S_n.
We have $V \cong D \oplus B$ as vector spaces. A simple calculation shows that this map is also an isomorphism of representations. $D \cong \mathbb{C}$ is irreducible, since $\dim(\mathbb{C}) = 1$. The representation B is also irreducible. We prove this in corollary 4.1.0.3.

Example 2.7.11. We continue with example 2.7.3, using the notation of example 2.7.10.

$$V_k = \bigwedge^k V \cong \bigwedge^k (D \oplus B) \cong \bigoplus_{l=0}^{k} \left((\wedge^l D) \bigotimes_{\mathbb{C}} (\wedge^{k-l} B) \right) = \wedge^{k-1} B \bigoplus \wedge^k B.$$

A basis of B is given by $\overline{v_i} := v_i + D$ with $1 \leq i \leq n-1$. Therefore $\wedge^k B$ is non zero for $0 \leq k \leq n-1$. We will prove in corollary 4.1.0.3 that those are irreducible.

One of the main objects in representation theory is the character.

Definition 2.7.12. *The character χ of a representation $\pi : G \times W \to W$ is defined as*

$$\chi(g) := \operatorname{tr}(\pi_g)$$

The character χ is called irreducible if W is irreducible.
Otherwise it is called reducible.

The most important characters we will use are:

Definition 2.7.13. *For $g \in S_n$ and $1 \leq k \leq n$ we set*

$$e_k^{(n)} = e_k^{(n)}(g) := \sum_{1 \leq i_1 < \cdots < i_k \leq n} t_{i_1} \cdots t_{i_k} \tag{2.7.5}$$

where the t_i are the eigenvalues of g (in any order).
For completeness we set $e_0^{(n)}(g) \equiv 1$ and $e_k^{(n)} \equiv 0$ for $k > n$.

If the dimension is clear, we will just write e_k.

Lemma 2.7.14. *The character of the representation $V_k^{(n)}$ is $e_k^{(n)}$.*

Proof. See [9], section 36. □

Remark: Please do not confuse $e_k^{(n)}$ with the character $\mathbf{e_k}$ used in the book *Lie Groups* by Bump [9]! We will see in lemma 4.1.0.2 that $e_k^{(n)}$ is only irreducible for $k = 0$ and $k = n$. Since all $\mathbf{e_k}$ are irreducible, they cannot be equal to $e_k^{(n)}$.

Characters have many important properties.

Lemma 2.7.15. *Let W_1 and W_2 be representations. Then $W_1 \oplus W_2$ and $W_1 \otimes_{\mathbb{C}} W_2$ are also representations with*

$$g(w_1 \oplus w_2) := (gw_1) \oplus (gw_2) \text{ and } g(w_1 \otimes w_2) := (gw_1) \otimes (gw_2) \quad \forall g \in G$$

The characters of $W_1 \oplus W_2$ and $W_1 \otimes W_2$ are

- $\chi_{W_1 \oplus W_2} = \chi_{W_1} + \chi_{W_2}$,
- $\chi_{W_1 \otimes W_2} = \chi_{W_1} \chi_{W_2}$.

Each character is the sum of irreducible characters.

The reason why characters are so important is that we can use them to decide if two (irreducible) representations are isomorphic or not.

Lemma 2.7.16. *Let W_1 and W_2 be irreducible representations with characters χ_1 and χ_2. Then*

$$\mathbb{E}\left[\chi_1 \overline{\chi_2}\right] = \begin{cases} 1, & \text{if } W_1 \cong W_2; \\ 0, & \text{otherwise.} \end{cases}$$

where $\mathbb{E}\left[f\right] = \mathbb{E}_G\left[f\right] := \frac{1}{|G|} \sum_{g \in G} f(g)$.
The characters are a basis in the space of class functions.

Remark: It is usual in representation theory to use the notation $\int_G (..)\, dg$ and not $\mathbb{E}\left[(..)\right]$. We do not follows this convention since it is very confusing switching always between $\int_G (..)\, dg$ and not $\mathbb{E}\left[(..)\right]$.

Corollary 2.7.16.1. *Let χ be an irreducible character.*

$$\mathbb{E}_G[\chi] = \begin{cases} 1, & \chi \equiv 1; \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 2.7.16.2. *Let W be a representation with character χ. Then*

$$\mathbb{E}_G[\chi] = \dim_{\mathbb{C}}(W^G)$$

with $W^G := \{w \in W; gw = g \quad \forall g \in G\}$

We write $\mathbb{E}\left[.\right]$ instead of $\mathbb{E}_G[.]$ if it is clear which group G is used.

One of the most important properties of the character is

Lemma 2.7.17. *Let W be a representation with character χ. Then the following are equivalent*

- *W is irreducible*
- $\mathbb{E}[\chi \overline{\chi}] = 1$

Chapter 3

Central limit theorem for associated class functions

This section is devoted to associated class functions on S_n and the proof of a central limit theorem for them.
We introduce in section 3.1 the definition of the characteristic polynomial $Z_n(x)$ and associated class functions $W^n(f)(x)$ on S_n. We also state a central limit theorem for the characteristic polynomial $Z_n(x)$ proven by B.M.Hambly, P.Keevash, N.O'Connell and D.Stark. In the rest of this section we then prove a central limit theorem for associated class functions. To do this, we first state in section 3.2 an auxiliary central limit theorem and then apply it in section 3.3 to associated class functions.

3.1 Definition and classical limit theorem

We first state the main result in the paper "*The characteristic polynomial of a random permutation matrix*" [17]. To do this, we need the following definition

Definition 3.1.1. *We set for $x \in \mathbb{C}, g \in S_n$*

$$Z_n(x) = Z_n(x)(g) = \det(I - xg). \tag{3.1.1}$$

where we interpret the elements of S_n as unitary matrices (see section 2.1.4).

Theorem 3.1.2 (B.M.Hambly, P.Keevash, N.O'Connell and D.Stark)**.** *Let $g \in S_n$ be chosen uniformly at random and x be a fixed complex number with $|x| = 1$, not a root of unity and of finite type. Then*

$$\operatorname{Re}\left(\frac{\log\big(Z_n(x)\big)}{\sqrt{\frac{\pi}{12}\log(n)}}\right) \xrightarrow{d} \mathcal{N}_1, \qquad \operatorname{Im}\left(\frac{\log\big(Z_n(x)\big)}{\sqrt{\frac{\pi}{12}\log(n)}}\right) \xrightarrow{d} \mathcal{N}_2 \tag{3.1.2}$$

with $\mathcal{N}_1, \mathcal{N}_2$ standard Gaussian random variables.

It is natural to ask if $\mathcal{N}_1$ and $\mathcal{N}_2$ are independent. This question is not considered in [17]. We will prove in corollary 3.1.6.1 that $\mathcal{N}_1, \mathcal{N}_2$ are indeed independent.

We extend in this section the ideas of paper "*The characteristic polynomial of a random permutation matrix*" [17] and prove a central limit theorem for associated class functions on S_n. The big difference between our work and [17] is that we calculate the correlation between the real and the imaginary part. Before we define associated class functions, we give a more explicit expression for $Z_n(x)(\sigma)$ using the cycle-type of σ. This is possible since $Z_n(x)$ is a class function and each conjugation class is uniquely determined by its cycle-type. We first look at the case $\sigma = (123\cdots n) \in S_n$. Then

$$g = g(\sigma) = (\delta_{i,\sigma(j)})_{1\le i,j\le n} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}. \tag{3.1.3}$$

Let $\alpha(m) := \exp(m\frac{2\pi i}{n})$. Then

$$g \begin{pmatrix} \alpha(m) \\ \alpha(m)^2 \\ \vdots \\ \alpha(m)^{n-1} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \alpha(m) \\ \vdots \\ \alpha(m)^{n-1} \\ \alpha(m)^{n-1} \end{pmatrix} = \overline{\alpha(m)} \begin{pmatrix} \alpha(m) \\ \alpha(m)^2 \\ \vdots \\ \alpha(m)^{n-1} \\ 1 \end{pmatrix}. \tag{3.1.4}$$

Therefore g is diagonalizable, the eigenvalues of g are $\left\{\overline{\alpha(m)}\right\} = \{\alpha(m)\}$ and $\det(I - xg) = 1 - x^n$.

The arbitrary case now follows easily. Fix an element $\sigma = \sigma_1\sigma_2\cdots\sigma_l \in S_n$ with σ_i disjoint cycles of length λ_i. Then

$$g = g(\sigma) = \begin{pmatrix} P_1 & & & \\ & P_2 & & \\ & & \ddots & \\ & & & P_l \end{pmatrix} \tag{3.1.5}$$

where P_i is a block matrix of a cycle as in (3.1.3) (after a possible renumbering of the basis). Then

$$Z_n(x)(g) = \prod_{i=1}^{l(\lambda)} (1 - x^{\lambda_i}). \tag{3.1.6}$$

In view of this equation we set

Definition 3.1.3. *Let $f : S^1 \to \mathbb{C}$ be given. We define*

$$W^n(f) = W^n(f)(x) = W^n(f)(x)(g) := \prod_{i=1}^{l(\lambda)} f(x^{\lambda_i}). \tag{3.1.7}$$

We call $W^n(f)$ the associated class function of f.

The function $W^n(f)$ only depends on the cycle-type of $g \in S_n$ (see section 2.1.1). Therefore $W^n(f)$ is a class function on S_n and $W^n(f)$ is independent of the interpretation of S_n as a subgroup of the unitary matrices. We use the expression of $W^n(f)$ in (3.1.7) in section 5, but in this section we need another formulation. We state this as a lemma.

Lemma 3.1.4. *Let $C_m^{(n)} = C_m^{(n)}(g)$ be the cycle numbers as in definition 2.1.7. Then*

$$Z_n(x) = \prod_{m=1}^{n} (1 - x^m)^{C_m^{(n)}}, \tag{3.1.8}$$

$$W^n(f)(x) = \prod_{m=1}^{n} f(x^m)^{C_m^{(n)}}. \tag{3.1.9}$$

Proof. This follows immediately from (3.1.6) since $C_m^{(n)}(g) = \#\{\lambda_i; \lambda_i = m\}$. □

It is clear that $Z_n(x) = W^n(1-x)$. The expressions in (3.1.8) and (3.1.9) are well defined since $C_m^{(n)} \in \mathbb{N}$. We emphasize this because we need later expressions of the form a^b with $b \in \mathbb{C}$.
We define in section 5 other class functions $W^{1,n}(f)$ and $W^{2,n}(f)$ on S_n. The definitions of $W^{1,n}(f)$ and $W^{2,n}(f)$ are very similar to the definition of $W^n(f)$. The main difference is that $W^{1,n}(f)$ and $W^{2,n}(f)$ have an additional randomization. $W^n(f)$ is in fact a special case of $W^{1,n}(f)$ and $W^{2,n}(f)$.

Before we can formulate the main result of this section, we have to define

Definition 3.1.5. *Let $f : S^1 \to \mathbb{C}$ be a real analytic function, $x \in S^1$ be arbitrary but fixed. We set*

$$auxm(f)(x) = \begin{cases} \int_0^1 \log\bigl(f(e^{2\pi is})\bigr)\, ds & \text{if } x \text{ is not a root of unity and the integral exists,} \\ \frac{1}{p}\sum_{m=1}^{p} \log\bigl(f(x^m)\bigr) & \text{if } x \text{ is a root of unity of order } p \text{ and the sum exists,} \\ \infty & \text{otherwise.} \end{cases} \tag{3.1.10}$$

We know that an analytic function $f \neq 0$ has only isolated zeros and $\log\bigl(f(e^{2\pi is})\bigr) \sim C\log(s-s_0)$ for $s \to s_0$ and s_0 a zero of $f(e^{2\pi is})$. Therefore the integral in the definition exists for all $f \neq 0$ and $m(f)(x) = \infty$ if and only if $f \equiv 0$ or x is a root of unity and $f(x^m) = 0$ for some $m \in \mathbb{N}$.
One can rewrite the integral in the definition of $m(f)$ as $\int_{S^1} \log\bigl(f(\varphi)\bigr)\, d\varphi$. We have not used this because this can be mistaken with the complex integral $\int_{S^1} \log\bigl(f(z)\bigr)\, dz = 2\pi i \int_0^1 \log\bigl(f(e^{2\pi is})\bigr) e^{2\pi is}\, ds$.
We now come to the main theorem.

Theorem 3.1.6. *Let $f : S^1 \to \mathbb{C}$ be real analytic and x be arbitrary and fixed. We define*

$$\log\left(W^n(f)(x)\right) := \sum_{m=1}^{n} C_m^{(n)} \log\bigl(f(x^m)\bigr) \tag{3.1.11}$$

with $\log$ *the principal branch of the logarithm and* $\log(-y) := \log(y) + i\pi$ *for* $y \in \mathbb{R}_{>0}$ *and* $\log(0) = \infty$. *We then have*

1. *If x is not a root of unity, of finite type and all zeros of f are roots of unity then*

$$\frac{\log\left(W^n(f)(x)\right)}{\sqrt{\log(n)}} - \sqrt{\log(n)}\, m(f)(x) \xrightarrow{d} \mathcal{N}_1 + i\mathcal{N}_2 \tag{3.1.12}$$

with $\mathcal{N}_1$ and $\mathcal{N}_2$ normal distributed random variables. The correlation of $\mathcal{N}_1$ and $\mathcal{N}_2$ is given by

$$\operatorname{Im}\left(\int_0^1 \log^2\bigl(f(e^{2\pi is})\bigr)\, ds\right). \tag{3.1.13}$$

The random variables $\mathcal{N}_1$ and $\mathcal{N}_2$ are independent if and only if $\operatorname{Im}\left(\int_0^1 \log^2\bigl(f(e^{2\pi is})\bigr)\, ds\right) = 0$.

2. *If x is a root of unity of order p and $f(x^m) \neq 0$ for all $1 \leq m \leq p$ then*

$$\frac{\log\left(W^n(f)(x)\right)}{\sqrt{\log(n)}} - \sqrt{\log(n)}m(f) \xrightarrow{d} \mathcal{N}_1 + i\mathcal{N}_2 \tag{3.1.14}$$

with $\mathcal{N}_1$ and $\mathcal{N}_2$ normal distributed random variables. The correlation of $\mathcal{N}_1$ and $\mathcal{N}_2$ is given by

$$\operatorname{Im}\left(\frac{1}{p}\sum_{m=1}^{p}\log^2\big(f(x^m)\big)\right) \tag{3.1.15}$$

The random variables $\mathcal{N}_1$ and $\mathcal{N}_2$ are independent if and only if $\operatorname{Im}\left(\frac{1}{p}\sum_{m=1}^{p}\log^2\big(f(x^m)\big)\right)$

Intuitively, the condition x of finite type ensures that the sequence $(x^m)_{m=1}^n$ does not approach too fast to the zeros of f for $n \to \infty$. This condition is essential in our proof.

As announced in the beginning, we can now prove

Corollary 3.1.6.1. *The random variables $\mathcal{N}_1$ and $\mathcal{N}_2$ in theorem 3.1.2 are independent.*

Proof. We know that $Z_n(x) = W^n(1-x)$. We can therefore apply theorem 3.1.6. We have to show that (3.1.13) is fulfilled. We show for completeness also that $m(1-x) = 0$. We have $\log(1-e^{2\pi i s}) = \log|1-e^{2\pi i s}| + i(\frac{1}{2}-s)\pi$ for $s \in [-\frac{1}{2}, \frac{1}{2}]$ Therefore

$$\begin{aligned} m(1-x) &= \int_0^1 \log(1-e^{2\pi i s})\ ds = \int_{-1/2}^{1/2} \log|1-e^{2\pi i s}| + i(\frac{1}{2}-s)\pi\ ds \\ &= \int_{-1/2}^{1/2} \log|1-e^{2\pi i s}|\ ds + i\int_{-1/2}^{1/2} (\frac{1}{2}-s)\pi\ ds \end{aligned}$$

The first integral can be computed with Jensen's formula (see lemma 2.2.4) and is also equal to 0. The second integral is 0 since the integrand is odd. Therefore $m(1-x) = 0$.

We next compute

$$\begin{aligned} \frac{1}{2}\operatorname{Im}\left(\int_0^1 \log^2(1-e^{2\pi i s})\ ds\right) &= \int_{-1/2}^{1/2} \operatorname{Re}\big(\log(1-e^{2\pi i s})\big)\operatorname{Im}\big(\log(1-e^{2\pi i s})\big)\ ds \\ &= \int_{-1/2}^{1/2} (\frac{1}{2}-s)\pi\log|1-e^{2\pi i s}|\ ds \\ &= \int_{-1/2}^{1/2} \frac{1}{2}\pi\log|1-e^{2\pi i s}|\ ds - \int_{-1/2}^{1/2} s\pi\log|1-e^{2\pi i s}|\ ds \end{aligned}$$

The first summand is up to a constant factor the same as the first summand above and is therefore equal to 0. The second integral is 0 since the integrand is odd. This proves the independence. □

We now proof theorem 3.1.6. Since this proof is not short, we split it into two parts and first prove a limit theorem.

3.2 Auxiliary central limit theorem

We need in this section the cycle lengths $C_m^{(n)}$ (see definition 2.1.7) and the Feller-coupling (see section 2.1.3). We now prove the following auxiliary central limit theorem

Theorem 3.2.1. *Let $(c_m)_{m=1}^\infty$ be a sequence of complex numbers with $a_m = \mathrm{Re}(c_m)$ and $b_m = \mathrm{Im}(c_m)$. We define and $A_n := \sum_{m=1}^n a_m C_m^{(n)}$ and $B_n := \sum_{m=1}^n b_m C_m^{(n)}$. If we have*

1. $|b_m| \leq 2\pi$
2. $\frac{1}{n}\sum_{m=1}^n |a_m| \to E_a$ *for* $n \to \infty$.
3. $\frac{1}{n}\sum_{m=1}^n a_m^2 \to V_a$, $\frac{1}{n}\sum_{m=1}^n b_m^2 \to V_b$
4. $\frac{1}{n}\sum_{m=1}^n |a_m|^3 = o\big(\log^{1/2}(n)\big)$.
5. $\frac{1}{n}\sum_{m=1}^n a_m b_m \to E_{ab}$ *for* $n \to \infty$.

then

$$\frac{A_n + iB_n - \mathbb{E}\left[A_n + iB_n\right]}{\sqrt{\log(n)}} \xrightarrow{d} \mathcal{N}_1 + i\mathcal{N}_2 \tag{3.2.1}$$

with $\mathcal{N}_1$ and $\mathcal{N}_2$ normal distributed random variables.
The correlation of $\mathcal{N}_1$ and $\mathcal{N}_2$ is E_{ab}. The random variables $\mathcal{N}_1$ and $\mathcal{N}_2$ are independent if and only if $E_{ab} = 0$.

Theorem 3.2.1 is the two dimensional analog of lemma 3.1 in [17]. The important difference is the calculation of correlation and the independence condition.

Proof. We use in this proof the Feller-coupling (see section 2.1.3). The random variables $C_m^{(n)}, C_m^{(n+1)}$ and Y_m are therefore defined on the same space.
We set $\widetilde{A}_n := \sum_{m=1}^n a_m Y_m$ and $\widetilde{B}_n := \sum_{m=1}^n b_m Y_m$. We use (2.1.12) and get

$$\begin{aligned}\frac{1}{\sqrt{\log(n)}}\mathbb{E}\left[\left|\widetilde{A}_n + i\widetilde{B}_n - A_n - iB_n\right|\right] &= \frac{1}{\sqrt{\log(n)}}\mathbb{E}\left[\left|\sum_{m=1}^n (a_n + ib_m)(Y_m - C_m^{(n)})\right|\right]\\ &\leq \frac{1}{\sqrt{\log(n)}}\left(\frac{2}{n+1}\sum_{m=1}^n \big(|a_m| + |b_m|\big)\right) \to 0\end{aligned} \tag{3.2.2}$$

by conditions (1) and (2). It follows that $\widetilde{A}_n + i\widetilde{B}_n$ has the same asymptotic behavior as $A_n + iB_n$. It is therefore enough to look at $\widetilde{A}_n + i\widetilde{B}_n$. We prove the theorem by calculating the asymptotic behavior of the characteristic function

$$\chi_n(t_a, t_b) := \mathbb{E}\left[\exp\left(\frac{it_a(\widetilde{A}_n - \mathbb{E}\left[\widetilde{A}_n\right]) + it_b(\widetilde{B}_n - \mathbb{E}\left[\widetilde{B}_n\right])}{\sqrt{\log(n)}}\right)\right]$$

for $|t_a|, |t_b| \leq K$ for some fixed $K \in \mathbb{R}_+$. We can calculate $\chi_n(t_a, t_b)$ explicitly since the random variables Y_m are independent and $\mathbb{E}\left[\exp(itY_m)\right] = \exp\left(\frac{e^{it}-1}{m}\right)$. We have

$$\begin{aligned}\chi_n(t_a, t_b) &= \exp\left(\sum_{m=1}^n \frac{e^{\frac{it_a a_m + it_b b_m}{\sqrt{\log(n)}}} - 1}{m}\right)\exp\left(\frac{-it_a\mathbb{E}\left[\widetilde{A}_n\right] - it_b\mathbb{E}\left[\widetilde{B}_n\right]}{\sqrt{\log(n)}}\right)\\ &= \exp\left(\sum_{m=1}^n \frac{e^{\frac{it_a a_m}{\sqrt{\log(n)}}}e^{\frac{it_b b_m}{\sqrt{\log(n)}}} - 1}{m}\right)\exp\left(\frac{-it_a\mathbb{E}\left[\widetilde{A}_n\right] - it_b\mathbb{E}\left[\widetilde{B}_n\right]}{\sqrt{\log(n)}}\right).\end{aligned} \tag{3.2.3}$$

We then write $e^{it} = \cos(t) + i\sin(t)$ and use the Taylor expansions of cos and sin. We have $\cos(t) = 1 - \frac{t^2}{2} + \frac{t^3}{6}\sin(\nu)$ with some $\nu \in [0,t]$ and thus $\cos(t) = 1 - \frac{t^2}{2} + O(t^3)$. A similar calculation holds for sin. We get

$$\cos\left(\frac{t_a a_m}{\sqrt{\log(n)}}\right) = 1 - \frac{t_a^2 a_m^2}{2\log(n)} + O\left(\frac{t_a^3 a_m^3}{\log^{3/2}(n)}\right), \tag{3.2.4}$$

$$\sin\left(\frac{t_a a_m}{\sqrt{\log(n)}}\right) = \frac{t_a a_m}{\sqrt{\log(n)}} + O\left(\frac{t_a^3 a_m^3}{\log^{3/2}(n)}\right) \tag{3.2.5}$$

where the big O's are uniformly in $m \in \mathbb{N}$. Similar formulas hold for b_m. We use these two identities together with (3.2.3) and obtain

$$\begin{aligned} e^{\frac{it_a a_m}{\sqrt{\log(n)}}} e^{\frac{it_b b_m}{\sqrt{\log(n)}}} - 1 =& \frac{1}{\sqrt{\log(n)}}(it_a a_m + it_b b_m) - \frac{1}{\log(n)}\left(\frac{a_m^2 t_a^2 + b_m^2 t_b^2}{2} - a_m b_m t_a t_b\right) \\ &+ \frac{1}{\log^{3/2}(n)} O\left(|a_m| + |a_m|^2 + |a_m|^3\right) \end{aligned}$$

since we have $|t_a|, |t_b| \le K$. We have on the other hand

$$\mathbb{E}\left[\widetilde{A}_n\right] = \sum_{m=1}^{n} \frac{a_m}{m} \text{ and } \mathbb{E}\left[\widetilde{B}_n\right] = \sum_{m=1}^{n} \frac{b_m}{m}$$

since the $\mathbb{E}[Y_m] = \frac{1}{m}$. Therefore

$$\begin{aligned} \chi_n(t_a, t_b) = \exp\Bigg(& \frac{1}{2\log(n)} \sum_{m=1}^{n} \frac{-(a_m^2 t_a^2 + b_m^2 t_b^2)}{m} + t_a t_b \frac{1}{\log(n)} \sum_{m=1}^{n} \frac{a_m b_m}{m} \\ &+ \frac{1}{\log^{1/2}(n)} O\left(\frac{1}{\log(n)} \sum_{m=1}^{n} \frac{|a_m| + |a_m|^2 + |a_m|^3}{m}\right)\Bigg). \end{aligned} \tag{3.2.6}$$

We apply lemma 2.2.3 to each summand. The first summand converges by condition (3) to $\frac{-1}{2}(V_a t_a^2 + V_b t_b^2)$. The second summand converges by condition (5) to E_{ab}. The third summand converges by lemma 2.2.3 and the conditions (2), (3) and (4) to 0. Therefore

$$\chi_n(t_a, t_b) \to e^{-\frac{V_a}{2} t_a^2 - \frac{V_b}{2} t_b^2 + E_{ab} t_a t_b}$$

point wise for all $|t_a|, |t_b| \le K$. Since K was arbitrary, $\chi_n(t_a, t_b)$ converge everywhere. We now can apply Lévy's continuity theorem (theorem 2.5.13). We set $t_b = 0$ to see that the real part converge to a normal distributed random variable. Similar for the imaginary part. It also follows directly from Lévy's continuity theorem that $\mathcal{N}_1$ and $\mathcal{N}_2$ are independent if and only if $E_{ab} = 0$. □

3.3 Proof of theorem 3.1.6

We are now ready to prove theorem 3.1.6. We recommend to read first section 2.3 and section 2.4 before reading this proof.

Proof of theorem 3.1.6. We have by definition $\log\Big(W^n(f)\Big) = \sum_{m=1}^{n} C_m^{(n)} \log\big(f(x^m)\big)$ (see (3.1.11)). We now apply theorem 3.2.1 with $c_m := \log\big(f(x^m)\big)$. To do this, we have to show that the conditions (1) - (5) are fulfilled.

Condition (1) is fulfilled by the definition of $\log\Big(W^n(f)\Big)$.

We show that conditions (2) - (5) are fulfilled by showing that all sums appearing there converge to a limit for $n \to \infty$.

We write $x = e^{2\pi it}$, $t_m := \{mt\}$, $\mathbf{t} = (t_m)_{m=1}^{\infty}$ and $h(s) := \big|\log\big|f(e^{2\pi is})\big|\big|$ and $b(s) := \arg\big(f(e^{2\pi is})\big)$. We have to distinguish several cases:
<u>Case 1.1</u>: x not a root of unity, $0 \notin f(S^1)$
We begin with condition (2). We have

$$\frac{1}{n}\sum_{m=1}^{n}|a_m| = \frac{1}{n}\sum_{m=1}^{n}|\mathrm{Re}(c_m)| = \frac{1}{n}\sum_{m=1}^{n}\Big|\log\big|f(x^m)\big|\Big| = \frac{1}{n}\sum_{m=1}^{n}h(t_m).$$

The function $h(s)$ is in this case a smooth function on $[0,1]$ and we can therefore apply theorem 2.3.2 to see that the last expression converges to $\int_0^1 h(s)\ ds$.
It follows with the same argument that

$$\frac{1}{n}\sum_{m=1}^{n}|a_m|^2 \to \int_0^1 h^2(s)\ ds \quad\text{and}\quad \frac{1}{n}\sum_{m=1}^{n}|a_m|^3 \to \int_0^1 h^3(s)\ ds$$

This proves condition (4) and the first part of condition (3).

We next look at $b_m = \mathrm{Im}\left(\log\big(f(x^m)\big)\right) = \arg(f(x^m)) = b(t_m)$. Since f is real analytic, one can use function theory to show that there exists a finite set $D \subset [0,1]$ such that $b(s)$ is real analytic in $[0,1] \setminus D$ and the limits $\lim_{s\uparrow s_0}\arg(f(e^{2\pi is}))$ and $\lim_{s\downarrow s_0}\arg(f(e^{2\pi is}))$ exists for all $s \in [0,1]$. We omit here the details since this is a standard argument.
The idea is now to split $[0,1]$ into subintervals such that $b(s)$ is real analytic in each subinterval and then to restrict the sequence $\mathbf{t}$ to each such subinterval. Lemma 2.3.11 shows that the restricted sequences are still uniformly distributed and we are therefore allowed to use theorem 2.3.2 on each subinterval separately. We thus obtain

$$\frac{1}{n}\sum_{m=1}^{n}b_m^2 \to \int_0^1 \big(b(s)\big)^2\ ds$$

and

$$\begin{aligned}\frac{1}{n}\sum_{m=1}^{n}a_m b_m \to \int_0^1 a(s)b(s)\ ds &= \int_0^1 \mathrm{Re}\left(\log\big(f(e^{2\pi is})\big)\right)\mathrm{Im}\left(\log\big(f(e^{2\pi is})\big)\right)\ ds\\ &= \frac{1}{2}\mathrm{Im}\left(\int_0^1 \log^2\big(f(e^{2\pi is})\big)\ ds\right)\end{aligned}$$

<u>Case 1.2.1</u> x not a root of unity and 1 is the only zero of f.
A simple calculation shows that $h(s) \sim C_1\log(s)$ and $\frac{d}{ds}h(s) \sim C_2\frac{1}{s}$ for $s \to 0$. Similarly for $s \to 1$. We therefore can not use anymore theorem 2.3.9. The idea is to replace theorem 2.3.2 by theorem 2.3.9.
To apply theorem 2.3.9, we have to know the behavior of $D_n([0,1],\mathbf{t})$ for $n \to \infty$ and to choose a "good" $\delta > 0$ with $\delta < t_m < 1-\delta$. This is the point where we need the finite type assumption of x. The behavior of $D_n([0,1],\mathbf{t})$ follows with this assumption from theorem 2.4.3. We have

$$D_n([0,1],\mathbf{t}) = O(n^{-\alpha}) \text{ for some } \alpha > 0 \tag{3.3.1}$$

Now we have to choose δ. Since x is of finite type, we can use lemma 2.4.2 to see that there exist constants $C>0,\beta>0$ such that

$$|mx-p|>\frac{C_1}{m^\beta} \text{ for all } m,p\in\mathbb{N}.$$

A simple calculation shows that $h(s)\sim C_2\log(s)$ and $\frac{d}{ds}h(s)\sim C_3\frac{1}{s}$ for $s\to 0$. Similar for $s\to 1$. We get with theorem 2.3.8

$$\begin{aligned}\left|\frac{1}{n}\sum_{m=1}^n h(t_m)-\int_0^1 h(t)\ ds\right|\leq&\left|\int_0^\delta h(s)\ ds\right|+\left|\int_{1-\delta}^1 h(s)\ ds\right| \qquad (3.3.2)\\ &+D_n([0,1],\mathbf{t})\int_\delta^{1-\delta}\frac{d}{ds}h(s)\ ds+O\left(\frac{C}{n^\beta}\log(n^\beta)\right)\\ \leq& O\left(n^{-\beta}\log(n^\beta)\right)+O\left(n^{-\alpha}\log(n^\beta)\right)+O\left(\frac{C}{n^\beta}\log(n^\beta)\right)\\ =&O(n^{-\gamma}) \text{ for some } \gamma>0.\end{aligned}$$

This shows that condition 2 is fulfilled. It is essential in (3.3.2) that $D_n([0,1],\mathbf{t})=O(n^{-\alpha})$. This shows that the finite type assumption is necessary for our proof.
A simple calculation shows that

$$\begin{aligned} h^2(s)&\sim C_4\log^2(s), & \frac{d}{ds}h^2(s)&\sim C_5\frac{\log(s)}{s},\\ h^3(s)&\sim C_6\log^3(s), & \frac{d}{ds}h^3(s)&\sim C_7\frac{\log^2(s)}{s}.\end{aligned}$$

It is now easy to see that one can use the calculations in (3.3.2) also for h^2 and h^3. This proves condition (4) and the first part of condition (3).
The argumentation for b_m is as above.

Case 1.2.2 x not a root of unity and all zeros of f occur at roots of unity.
Choose a fixed $q\in\mathbb{N}$ such that all zeros of f can be written as $\exp(2\pi i\frac{p}{q})$ with $p\in\mathbb{N}$. This is possible since by assumption all zeros of f are roots of unity and f is real analytic.
We set $I_p:=[\frac{p-1}{q},\frac{p}{q}]$ for $1\leq p\leq q$ and apply Theorem 2.3.9 separately to each I_p for the sequences $\mathbf{y}^{(p)}=\mathbf{x}\cap I_p$. We know from lemma 2.3.11 that $D_n(I_p,\mathbf{y}^{(p)})=O(n^{-\alpha})$ since $D_n([0,1],\mathbf{x})=O(n^{-\alpha})$. We can choose, as above with lemma 2.4.2, a $\delta=\frac{C}{n^\beta}$ with $\frac{p-1}{q}+\delta<y_m^{(p)}<\frac{p}{q}-\delta$ for $1\leq m\leq n$. This shows that the calculations are similar to the calculations in Case 1.2.1.

Case 2: x a root of unity of order p and $f(x^m)\neq 0$ for all $1\leq m\leq p$.
We have

$$\begin{aligned}\frac{1}{n}\sum_{m=1}^n|a_m|=&\frac{1}{n}\sum_{m=1}^{}h(t_m)=\sum_{k=1}^p\sum_{j=0}^{\left[\frac{n-k}{p}\right]}\frac{1}{jp+k}h(t_{jp+k})\\ =&\sum_{k=1}^p h(t_k)\left(\frac{1}{\log(n)}\sum_{j=0}^{\left[\frac{n-k}{p}\right]}\frac{1}{jp+k}\right). \qquad (3.3.3)\end{aligned}$$

It is easy to see that

$$\lim_{n\to\infty}\frac{1}{\log(n)}\sum_{j=0}^{\left[\frac{n-k}{p}\right]}\frac{1}{jp+k}=\lim_{n\to\infty}\frac{1}{\log(n/p)+\log(p)}\sum_{j=0}^{\left[\frac{n}{p}\right]}\frac{1}{jp}=\frac{1}{p}.$$

We have therefore proven that (2) is fulfilled.
The other calculations are similar. We therefore omit them.
We have until now proven that

$$\frac{\log\Big(W^n(f)\Big) - \mathbb{E}\left[\log\Big(W^n(f)\Big)\right]}{\sqrt{\log(n)}} \xrightarrow{d} \mathcal{N}_1 + i\mathcal{N}_2 \tag{3.3.4}$$

in all cases mentioned in theorem 3.1.6, including the correlation and the independence condition.

To complete the proof, we have to show that $\frac{\mathbb{E}\left[\log\left(W^n(f)\right)\right]}{\sqrt{\log(n)}} - \sqrt{\log(n)}m(f) \to 0$. We use lemma 2.1.8 and lemma 2.2.2 to see that

$$\mathbb{E}\left[\log\Big(W^n(f)\Big)\right] = \sum_{m=1}^{n} \frac{\log\big(f(x^m)\big)}{m} = \left(\frac{1}{n}\sum_{m=1}^{n}\log\big(f(x^m)\big)\right) + \int_1^n A(s)\frac{1}{s^2}\,ds \tag{3.3.5}$$

with $A(s) = \sum_{m\leq s}\log\big(f(x^m)\big)$.

If x is a root of unity, we use (3.3.2) to see that $\frac{1}{n}\sum_{m=1}^{n}\log\big(f(z^m)\big) = m(f) + O(n^{-\beta})$ for some $\beta > 0$ and therefore $A(s) = sm(f) + O(s^{1-\beta})$. We get with (3.3.5)

$$\begin{aligned}
\frac{1}{\sqrt{\log(n)}}\sum_{m=1}^{n}\frac{\log\big(f(z^m)\big)}{m} =& O\left(\frac{1}{\sqrt{\log(n)}}\right) + \frac{1}{\sqrt{\log(n)}}\int_1^n s\big(m(f) + O(s^{1-\beta})\big)\frac{1}{s^2}\,ds \\
=& O\left(\frac{1}{\sqrt{\log(n)}}\right) + \frac{m(f)}{\sqrt{\log(n)}}\int_1^n \frac{1}{s}\,ds + \frac{1}{\sqrt{\log(n)}}\int_1^n O(s^{-1-\beta})\,ds \\
=& \sqrt{\log(n)}m(f) + O\left(\frac{1}{\sqrt{\log(n)}}\right).
\end{aligned}$$

In this case x a root of unity, one has to replace (3.3.2) with $\sum_{m=1}^{n}\frac{1}{m} = \log(n) + C_8 + O\left(\frac{1}{n}\right)$ (see [3]). We omit the details since these calculations are similar to the ones above. □

Chapter 4

Calculation of $\mathbb{E}\left[Z_n^s(x)\right]$ with representation theory

In this section, we interpret S_n as a subgroup of $U(n)$ (see section 2.1.4) and use representation theory to write down a generating function for $\mathbb{E}\left[Z_n^s(x)\right]$ with respect to the uniform measure on S_n. We use this generating function to calculate the behavior of $\mathbb{E}\left[Z_n^s(x)\right]$ for $s \in \mathbb{N}$. These results are the same as in my paper [27], but obtained with a different method. We use in [27] the combinatorial argument of section 5, since this method is shorter and simpler to understood.

Most of the calculations in this section are based on the examples in section 2.7. We therefore recommend to take a look at the examples in section 2.7 before reading this section.
We have by definition

$$Z_n(x)(g) = \det(I - xg) = \prod_{k=1}^{n}(1 - xt_i) = \sum_{k=0}^{n}(-1)^k e_k(g)x^k \tag{4.0.1}$$

with $g \in S_n$ arbitrary and $t_1, \cdots, t_n$ the eigenvalues of g.
We know from (3.1.4) that all $g \in S_n$ can be diagonalized. This proves the second equality in (4.0.1). Third equality follows by expanding the product, collecting the coefficients of each x^k and then using the definition of $e_k(g)$ in (2.7.5). More details can be found in [9], in section 35.

We fix $s \in \mathbb{N}$ and rewrite $Z_n^s(x)$ as $\prod_{k=1}^{s} Z_n(\mathrm{x}_k)$ with $\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_s$ complex variables. We do this because we need it in section 4.4. We simplify the notation by setting

Definition 4.0.1. *Let $\mu = (\mu_1, \cdots, \mu_s) \in \mathbb{N}^s$ be arbitrary and $\mathrm{x}_1, \cdots, \mathrm{x}_s$ be complex variables. Then*

$$\mathbf{x}^\mu := \mathrm{x}_1^{\mu_1}\mathrm{x}_2^{\mu_2}\cdots\mathrm{x}_s^{\mu_s}, \quad l(\mu) := \#\left\{1 \le i \le n; \mu_i \ne 0\right\}, \quad |\mu| := \sum_{i=1}^{s}\mu_i. \tag{4.0.2}$$

We now get with (4.0.1)

$$\mathbb{E}\left[\prod_{k=1}^{s} Z_n(\mathrm{x}_k)\right] = \sum_{\mu \in \mathbb{N}^s} \mathbb{E}\left[e_\mu^{(n)}\right](-1)^{|\mu|}\mathbf{x}^\mu \tag{4.0.3}$$

with $e_\mu := e_{\mu_1}\cdots e_{\mu_s}$. Since $\mathbb{E}\left[(..)\right] = \frac{1}{n!}\sum_{g\in S_n}(..)$ is a finite sum and $e_k \equiv 0$ for $k > n$, there is no problem with the convergence or reordering in (4.0.3).
In view of the last equality, it is natural to ask: What is $\mathbb{E}\left[e_\mu\right]$?
We will take first a look at the case $l(\mu) = 2$.

4.1 Case $l(\mu) = 2$

The main result of this subsection is

Proposition 4.1.0.1. *We have* $\overline{Z_n(x)} = Z_n(\overline{x})$, $e_k(g) \in \mathbb{Z}$ *for all* $g \in S_n$ *and*

$$\mathbb{E}\left[e_k e_l\right] = \begin{cases} 2, & k = l \text{ and } 0 < k < n, \\ 1, & k = l \text{ and } 0 = k \text{ or } k = n, \\ 1, & k = l \pm 1 \text{ and } 0 \leq k, l \leq n, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1.1}$$

Before we prove this lemma, we give some corollaries.

Corollary 4.1.0.1. *We have*

$$\mathbb{E}\left[e_k\right] = \begin{cases} 1, & k = 0 \text{ or } k = 1. \\ 0, & \text{otherwise.} \end{cases} \tag{4.1.2}$$

$$\mathbb{E}\left[Z_n(x)\right] = 1 - x \tag{4.1.3}$$

Proof. This follows directly from proposition 4.1.0.1 and (4.0.3) with $e_0 \equiv 1$. □

Corollary 4.1.0.2. *The character* $e_k^{(n)}$ *is irreducible for* $k = 0$ *and* $k = n$.
The character $e_k^{(n)}$ *is the sum of two different irreducible characters for* $0 < k < n$.

Proof. We know from proposition 4.1.0.1 that

$$\mathbb{E}\left[e_k^{(n)}\overline{e_k^{(n)}}\right] = \mathbb{E}\left[e_k^{(n)}e_k^{(n)}\right] = \begin{cases} 2 & \text{for } 0 < k < n, \\ 1 & \text{for } k = 0 \text{ or } k = n. \end{cases}$$

We know from lemma 2.7.14 that $e_k^{(n)}$ is a character and therefore can be written by lemma 2.7.15 as

$$e_k^{(n)} = \sum_{j=1}^{J} m_j \chi_j$$

with $m_j \in \mathbb{N}$, χ_j irreducible characters and $\chi_i \neq \chi_j$ for $i \neq j$. We now use lemma 2.7.16 and get

$$\mathbb{E}\left[e_k^{(n)}\overline{e_k^{(n)}}\right] = \sum_{1 \leq i,j \leq J} m_i m_j \mathbb{E}\left[\chi_i \overline{\chi_j}\right] = \sum_{j=1}^{J} m_j^2 = \begin{cases} 2 & \text{for } 0 < k < n, \\ 1 & \text{for } k = 1 \text{ or } k = n. \end{cases}$$

Since $m_j \in \mathbb{N}$, the only possibilities to fulfill this equation are $j = 2$ and $m_1 = m_2 = 1$ for $0 < k < n$ and $j = 1, m_1 = 1$ for $k = 1$ or $k = n$. This proves the lemma. □

One might ask: What are the irreducible representations appearing in $e_k^{(n)}$?
We have shown in example 2.7.11

$$V_k = \wedge^{k-1} B \bigoplus \wedge^k B$$

Corollary 4.1.0.3. *The representations* $\wedge^k B$ *are irreducible for* $0 \leq k < n$.

Proof. We have $\wedge^0 B = \mathbb{C}$ by definition. The case $k = 0$ is therefore trivial.
We have found for $0 < k < n$ a non trivial decomposition of e_k into two factors. Since e_k is the sum of two irreducible factors, both have to be irreducible. □

We give more details on the decomposition into irreducible factors in section 4.2 and prove now proposition 4.1.0.1.

Proof of proposition 4.1.0.1. We first prove $e_k(g) \in \mathbb{Z}$.
We have $e_0 \equiv 1$ and $e_k \equiv 0$ for $k > n$, so we only have to look at $1 \leq k \leq n$.
We know form lemma 2.7.14 that e_k is the character of the representation V_k. Let $I \in \mathrm{SeT}_k$ be arbitrary. We have shown in example 2.7.3 that

$$gv_I = v_{\sigma(I)} \text{ or } gv_I = -v_{\sigma(I)} \text{ for } g = (\delta_{i,\sigma(j)})_{1\leq i,j\leq n} \in S_n$$

Because $e_k(g)$ is the trace of the automorphism given by the operation with g, $e_k(g)$ is a sum with all summands in $\{-1, 0, 1\}$.
To prove $\overline{Z_n(x)} = Z_n(\overline{x})$, we use (4.0.1)

$$\overline{Z_n(x)} = \overline{\sum_{k=0}^{n} (-1)^k e_k(g) x^k} = \sum_{k=0}^{n} (-1)^k \overline{e_k(g)}\,\overline{x}^k = Z_n(\overline{x}).$$

We know from lemma 2.7.14 and lemma 2.7.15 that $e_k e_l$ is the character of the representation $\wedge^k V \bigotimes_{\mathbb{C}} \wedge^l V$. Therefore we set

Definition 4.1.1. *For $k, l \in \mathbb{N}$ we set*

$$V_{k,l} = V_{k,l}^{(n)} := \wedge^k V \bigotimes_{\mathbb{C}} \wedge^l V. \tag{4.1.4}$$

To write down a basis of $V_{k,l}$, we need

Definition 4.1.2. *We define $\mathrm{SeT}_{k,l} = \mathrm{SeT}_{k,l}^{(n)}$ to be the set of all ordered pairs (I_1, I_2) with $I_i \subset \{1, \cdots, n\}$ and $|I_1| = k, |I_2| = l$.*

We set

$$v_I := v_{I_1} \otimes v_{I_2} \qquad \text{for } I = (I_1, I_2) \in \mathrm{SeT}_{k,l}.$$

The collection $\{v_I\}_{I\in\mathrm{SeT}_{k,l}}$ forms a basis of $V_{k,l}$. The operation of $g = (\delta_{i,\sigma(j)})_{1\leq i,j\leq n} \in S_n$ on v_I is given by

$$gv_I = g(v_{I_1} \otimes v_{I_2}) = (gv_{I_1}) \otimes (gv_{I_2}) = v_{\sigma(I_1)} \otimes v_{\sigma(I_2)}.$$

Remark: we have

$$V_{k,0} = V_k \bigotimes_{\mathbb{C}} \mathbb{C} \cong V_k$$

and $\mathrm{SeT}_{k,0}$ is canonically bijective to SeT_k.
The notation for length $l(\mu) = 2$ is therefore compatible with the notation for length $l(\mu) = 1$ and so it is not necessary to distinguish the cases $l = 2$ and $l = 1$ in the proof.

We now look at the integral. We get with corollary 2.7.16.2

$$\mathbb{E}[e_k e_l] = \dim_{\mathbb{C}}\Big((V_{k,l})^{S_n}\Big)$$

where $(V_{k,l})^{S_n} := \{v \in V_{k,l}; gv = v\ \forall g \in G\}$. We write for shortness $\sum_I := \sum_{I\in\mathrm{SeT}_{k,l}}$.
Choose any $v \in (V_{k,l})^{S_n}$ with $v = \sum_I a_I v_I$. Then

$$v = \frac{1}{n!} \sum_{g\in S_n} gv = \sum_I a_I \left(\frac{1}{n!} \sum_{g\in S_n} g(v_I) \right). \tag{4.1.5}$$

We set $T(v_I) := \frac{1}{n!} \sum_{g\in S_n} g(v_I)$. Therefore $(V_{k,l})^{S_n}$ is generated by the vectors of the form $T(v_I)$. For this reason, we have to look at the vectors $T(v_I)$ only.

Lemma 4.1.3. *T has the following properties:*

1. *T is linear.*
2. *Let $g \in S_n$ be arbitrary. Then*
$$gT(v_I) = T(v_I) = T(gv_I)$$
3. *We write $T(v_I) = a_I v_I + \sum_{J \neq I} a_J v_J$. Then*
$$T(v_I) = 0 \Longleftrightarrow a_I = 0$$
4. *Choose any $v_{I_1} \otimes v_{I_2}$ fix with $T(v_{I_1} \otimes v_{I_2}) \neq 0$.*
   It is equivalent
   - *$T(v_{J_1} \otimes v_{J_2}) = \pm T(v_{I_1} \otimes v_{I_2})$*
   - *It exists a $\sigma \in S_n$ with $\sigma(I_i) = J_i$ for $1 \leq i \leq 2$*

Proof. (1) and (2) are obvious.
(3) We have

$$T(v_I) = gT(v_I) = g\Big(a_I v_I + \sum_{J \neq I} a_J v_J\Big) = a_I g v_I + \sum_{J \neq I} a_J g v_J$$

Therefore $a_I = 0$ is equivalent to the fact, that no basis vector of the from gv_I appears in $T(v_I)$. By the definition of $T(v_I)$, only such basis vectors can appear. Therefore $T(v_I) = 0 \Longleftrightarrow a_I = 0$.
(4) " $\Longleftarrow$ " follows from (2)
" $\Longrightarrow$ " Since $T(v_{I_1} \otimes v_{I_2}) \neq 0$, $v_{I_1} \otimes v_{I_2}$ appears in $T(v_{I_1} \otimes v_{I_2}) \neq 0$, it therefore also appears in $T(v_{J_1} \otimes v_{J_2})$. This is only possible if there exists a $g \in S_n$ with $g(v_{J_1} \otimes v_{J_2}) = \pm v_{I_1} \otimes v_{I_2}$ □

In light of point (4), we define

Definition 4.1.4. *We call $I \in \mathrm{SeT}_{k,l}$ and $J \in \mathrm{SeT}_{k,l}$ equivalent if there exists $\sigma \in S_n$ with $\sigma(I_i) = J_i$ for $1 \leq i \leq 2$. In this case we write $I \sim J$.*

It is easy to see that $\sim$ is an equivalence relation. We know form lemma 4.1.3 that $T(v_I) = \pm T(v_J)$ exactly when $I \sim J$.
Let F be a set representatives of $\mathrm{SeT}_{k,l}/\sim$.

Lemma 4.1.5. *The non zero vectors $T(v_I)$ with $I \in F$ are linearly independent.*

Proof. Let $\sum_{I \in F} b_I T(v_I) = 0$ for some b_I. Suppose there exists an $I \in F$ with $b_I \neq 0$ and $T(v_I) \neq 0$.
Since v_I appears in $T(v_I)$ by lemma 4.1.3 and all v_I are linearly independent, v_I has to appear in another $T(v_J)$ with $J \neq I$ as well. But this is only possible if $I \sim J$. This is a contradiction. □

Therefore it is enough to find all I for which $T(v_I) \neq 0$ (modulo equivalence).

Lemma 4.1.6.

$$T(v_{I_1} \otimes v_{I_2}) \neq 0 \Longleftrightarrow |I_1 \setminus I_2| \leq 1 \text{ and } |I_2 \setminus I_1| \leq 1.$$

Proof. Let $v_{I_1} \otimes v_{I_2}$ be given. We use lemma 4.1.3.3 to determine when $T(v_{I_1} \otimes v_{I_2}) = 0$. Since S_n permutes the bases $v_{I_1} \otimes v_{I_2}$ (with a possible change of sign), we have to calculate

$$\begin{aligned} \sum_{\substack{g\in S_n \\ g(v_{I_1}\otimes v_{I_2})=\pm v_{I_1}\otimes v_{I_2}}} g(v_{I_1} \otimes v_{I_2}) &= \sum_{\substack{\sigma\in S_n \\ \sigma(I_1)=I_1,\sigma(I_2)=I_2}} g(v_{I_1} \otimes v_{I_2}) \\ &= \sum_{\substack{\sigma\in S_n \\ \sigma(I_1\setminus I_2)=I_1\setminus I_2 \\ \sigma(I_2\setminus I_1)=I_2\setminus I_1 \\ \sigma(I_1\cap I_2)=I_1\cap I_2}} g(v_{I_1} \otimes v_{I_2}). \end{aligned} \tag{4.1.6}$$

Choose a $\sigma \in S_n$, which permutes $I_1 \setminus I_2$ and leaves the rest fixed. Then $gv_{I_1} = \epsilon(\sigma)v_{I_1}$ and $gv_{I_2} = v_{I_2}$. Therefore $g(v_{I_1} \otimes v_{I_2}) = \epsilon(\sigma)v_{I_1} \otimes v_{I_2}$
If $\sigma \in S_n$ permutes $I_1 \cap I_2$ and leaves the rest fixed, then we get on both factors a -1 or a 1 and therefore $g(v_{I_1} \otimes v_{I_2}) = v_{I_1} \otimes v_{I_2}$.
We write $Per(I_1 \setminus I_2)$ for all permutations of the set $I_1 \setminus I_2$ and get

$$\sum_{\substack{g\in S_n \\ g(v_{I_1}\otimes v_{I_2})=\pm v_{I_1}\otimes v_{I_2}}} g(v_{I_1} \otimes v_{I_2}) = \text{Const.} \sum_{\substack{\sigma_1\in Per(I_1\setminus I_2) \\ \sigma_2\in Per(I_2\setminus I_1)}} \epsilon(\sigma_1)\epsilon(\sigma_2)v_{I_1} \otimes v_{I_2}.$$

But $\sum_{\sigma\in S_m} \epsilon(\sigma) = 0$ if $m > 1$. □

Corollary 4.1.6.1. *The dimension* $\dim_{\mathbb{C}}\left((V_{k,l})^{S_n}\right)$ *is equal to the number of non equivalent sets* $(I_1, I_2) \in \mathrm{SeT}^n_{(k,l)}$ *with* $|I_1 \setminus I_2| \le 1$ *and* $|I_2 \setminus I_1| \le 1$

Proof. We know that $(V_{k,l})^{S_n}$ is generated by the vectors $T(v_I)$ with $I \in \mathrm{SeT}_{k,l}$. The corollary follows now from lemma 4.1.5 and lemma 4.1.6. □

We can now finish the proof of proposition 4.1.0.1.
Recall: $\mathbb{E}\left[e_k e_l\right] = \dim_{\mathbb{C}}\left((V_{k,l})^{S_n}\right)$. We can use corollary 4.1.6.1 to calculate this integral.
W.l.o.g.: $k \ge l$ and $I_1 = \{1, \cdots, k\}$.
Since $|I_1 \setminus I_2| \le 1$ only the cases $k - 1 = l$ and $k = l$ are non trivial.
Case 1: $k - 1 = l$
There is only one relevant possibility: $I_2 = \{1, \cdots, k-1\}$, since the cases $I_2 = \{1, \cdots, k\} \setminus \{s\}$ for $1 \le s \le k$ are all equivalent!
Case 2: $k = l$
In this situation we have two possibilities: Either $I_1 = I_2$ or $I_2 = I_1 \cup \{p\} \setminus \{q\}$ with $1 \le p \le k < q$. The second case is only possible, if $k < n$, since I_2 has a point outside I_1.

We have therefore proven for $0 \le l \le k \le n$ that

$$\mathbb{E}\left[e_k e_l\right] = \begin{cases} 2, & \text{if } k = l \text{ and } 0 < k < n; \\ 1, & \text{if } k = l \text{ and } k = 0 \text{ or } k = n; \\ 1, & \text{if } k = l + 1 \text{ and } 0 < k \le n; \\ 0, & \text{otherwise.} \end{cases}$$

□

4.2 Decomposition of $Z_n(x)$ into irreducible characters

Since $Z_n(x)$ is a class functions we can write it as a sum of irreducible characters. It is natural to ask if we can decide which irreducible representations appear in $Z_n(x)$. For this, we need some theorems from the book "*Lie groups*" [9]. We state them here very shortly and refer for more details to [9]. We do this since the stated theorems are used in this subsection only and it would need much more time and space to give further details.

Definition 4.2.1. *Let λ, ν be partitions. We write $\nu \subset \lambda$ if $\nu_i \leq \lambda_i$.*

Theorem 4.2.2. *The irreducible representations of S_n can be parameterized by partitions of n. We write ω_λ for the representation corresponding to the partition λ and s_λ for the character of ω_λ.*
We have $1 = s_{(n)}$ and $\epsilon = s_{(1^n)}$ for all $n \in \mathbb{N}$ and $(1^n) := \underbrace{(1, 1, \cdots, 1)}_{n \text{ times}}$.

Proof. See [9]. □

We calculate the decomposition of $Z_n(x)$ into irreducible characters by induction over n. To do this, we embeds $S_{n-1} \hookrightarrow S_n$ with $\sigma(n) := n$ and write $V^{(n)}$ and $B^{(n)}$ for V and B to emphasize the dimension. The main theorem we need is

Theorem 4.2.3. *Let λ be a partition of n. Then*

$$s_\lambda\Big|_{S_{n-1}} = \sum_{\substack{\nu \\ \nu \subset \lambda, |\nu| = n-1}} s_\nu \tag{4.2.1}$$

Proof. See [9], section 42. □

Lemma 4.2.4. *We have*

$$B^{(n)}\Big|_{S_{n-1}} \cong V^{(n-1)} \text{ as representations of } S_{n-1}. \tag{4.2.2}$$

Proof. We write $b_1, \cdots, b_n$ for the canonical projection of $v_1, \cdots, v_n$ on $B^{(n)} = V^{(n)}/D$. We have that $gb_i = b_{\sigma(i)}$ for all $g = \sigma \in S_n$. This follows from the definition of the action of S_n on $V^{(n)}$. The action of S_n on $B^{(n)}$ looks very similar to the action on $V^{(n)}$. The difference is that $b_1, \cdots, b_n$ is not a basis of $B^{(n)}$. The vectors $b_1, \cdots, b_{n-1}$ are a basis, but g does not necessary map this basis to itself. g maps this basis to itself if and only if $\sigma(n) = n$. We get that $B^{(n)}\big|_{S_{n-1}} \cong V^{(n-1)}$, since the action of S_{n-1} on the basis $b_1, \cdots, b_{n-1}$ of $B^{(n)}$ and on the basis $v_1, \cdots, v_{n-1}$ of $V^{(n-1)}$ are the same. □

We can now prove

Theorem 4.2.5. *We have for $0 \leq k \leq n-1$*

$$\wedge^k B^{(n)} = \chi_{(n-k, 1^k)} \tag{4.2.3}$$

and

$$Z_n(x) = (1-x) \sum_{k=1}^{n-1} s_{(n-k, 1^k)}. \tag{4.2.4}$$

Proof. We write $b(k,n)$ for the character of $\wedge^k B^{(n)}$. We know from corollary 4.1.0.3 that $\wedge^k B^{(n)}$ is an irreducible representation of S_n. It follows from theorem 4.2.2 that there exists a partition λ such that $b(k,n) = s_\lambda$. We use theorem 4.2.3 and a double induction over k and n to determine λ.
$\underline{k=0}$:
This case is trivial since we have by definition $\wedge^0 B^{(n)} = \mathbb{C}$ and $1 = s_{(n)}$.
$\underline{k=1}$:
We use lemma 4.2.4 and $V \cong B \oplus D$ (see example 2.7.10) to get

$$B^{(n)}\Big|_{S_{n-1}} \cong V^{(n-1)} \cong B^{(n-1)} \oplus D = B^{(n-1)} \oplus \mathbb{C}. \tag{4.2.5}$$

Therefore $b(1,n)\Big|_{S_{n-1}} = b(1,n-1)+1 = b(1,n-1)+s_{(n-1)}$. We can now use theorem 4.2.3. It is obvious that the only two partitions of n with $n-1 \subset \lambda$ are $\lambda = (n)$ and $\lambda = (n-1,1)$. We cannot have $b(1,n) = s_{(n)}$ since $s_{(n)}\big|_{S_{n-1}} = s_{(n-1)}$ is not the sum of two irreducible characters. Therefore we must have $b(1,n) = s_{(n-1,1)}$.
$\underline{k-1 \to k}$:
Here we use $\bigwedge^k V \cong \wedge^{k-1} B \bigoplus \wedge^k B$ (see example 2.7.11). We get

$$\bigwedge^k B^{(n)}\Bigg|_{S_{n-1}} \cong \bigwedge^k V^{(n-1)} \cong \wedge^{k-1} B^{(n-1)} \bigoplus \wedge^k B^{(n-1)}. \tag{4.2.6}$$

It follows by the induction hypothesis that

$$b(k,n)\Big|_{S_{n-1}} = b(k,n-1) + b(k-1,n-1) = s_{(n-1-k,1^k)} + s_{(n-k,1^{k-1})}. \tag{4.2.7}$$

It is obvious that the only partition λ of n with $(n-1-k,1^k) \subset \lambda$ and $(n-k,1^{k-1}) \subset \lambda$ is $\lambda = (n-k,1^k)$. This proves (4.2.3).
It remains to show (4.2.4). We have that $e_k^{(n)} = b(k,n)+b(k-1,n)$ since e_k is the character of $V_k^{(n)} = \bigwedge^k V^{(n)}$. We use (4.0.1) and get

$$\begin{aligned} Z_n(x) &= \sum_{k=0}^{n} (-1)^k e_k x^k = \sum_{k=0}^{n} (-1)^k (b(k,n) + b(k-1,n)) x^k \\ &= \sum_{k=0}^{n-1} (-1)^k b(k,n) + \sum_{k=1}^{n-1} (-1)^{k+1} b(k,n) x^{k+1} = (1-x) \sum_{k=0}^{n-1} (-1)^k b(k,n). \end{aligned} \tag{4.2.8}$$

We have used that $b(n,n) = 0$ since $\dim_{\mathbb{C}} B^{(n)} = n-1$ and therefore $\wedge^n B^{(n)} = 0$. $\square$

4.3 Case $l(\mu) > 2$

In this subsection we try to calculate $\mathbb{E}\left[e_\mu^{(n)}\right]$ for arbitrary μ. Unfortunately we cannot give an explicit expression as for $l(\mu) \leq 2$. We will instead reformulate $\mathbb{E}\left[e_\mu^{(n)}\right]$ as a number of solutions of a (simple) inequality system (see (4.3.5)).

We fix in this subsection $s \in \mathbb{N}$ and $\mu = (\mu_1, \mu_2, \cdots, \mu_s) \in \mathbb{N}^s$.
We will adapt the proof of proposition 4.1.0.1 to calculate $\mathbb{E}[e_\mu]$. For this, we first extend the definitions 4.1.2 and 4.1.4 to an arbitrary $\mu \in \mathbb{N}^s$.

Definition 4.3.1.

$$V_\mu = V_\mu^{(n)} := \wedge^{\mu_1} V \bigotimes_{\mathbb{C}} \cdots \bigotimes_{\mathbb{C}} \wedge^{\mu_s} V \tag{4.3.1}$$

with $\wedge^0 V := \mathbb{C}$.

Definition 4.3.2. *We define* $\mathrm{SeT}_\mu = \mathrm{SeT}_\mu^{(n)}$ *to be the set of all sequences* $(I_i)_{1\le i\le s}$ *with* $I_i \subset \{1,\cdots,n\}$ *and* $|I_i| = \mu_i$ *for* $1 \le i \le s$.

Definition 4.3.3. *We call* $I \in \mathrm{SeT}_\mu$ *and* $J \in \mathrm{SeT}_\mu$ *equivalent, if there exists a* $\sigma \in S_n$ *with* $\sigma(I_i) = J_i$ *for* $1 \le i \le s$. *In this case we write* $I \sim J$.

As in the proof of proposition 4.1.0.1 we need to know when $T(v_I) \neq 0$ for a given $I \in \mathrm{SeT}_\mu$. To do this, we introduce some terminology.

Definition 4.3.4. *For* $I \in SeT_\mu$ *and* $\epsilon \in \{1,c\}^s$, *we set*

$$I^\epsilon := I_1^{\epsilon_1} \cap \cdots \cap I_s^{\epsilon_s} \subset \{1,\cdots,n\}$$

with $I_i^1 = I_i$ *and* $I_i^c = \{1,\cdots,n\} \setminus I_i$.

Definition 4.3.5. *We call* $\epsilon \in \{1,c\}^s \left\{ \begin{array}{l} even \\ odd \end{array} \right.$ *if the number of* $1's$ *in* ϵ *is* $\left\{ \begin{array}{l} even \\ odd \end{array} \right.$.

The analog of corollary 4.1.6.1 is

Definition 4.3.6.

$$\mathfrak{N}_n(\mu) := \{I \in \mathrm{SeT}_\mu ; |I^\epsilon| \le 1 \text{ for all odd } \epsilon \in \{1,c\}^s\} \big/ \sim, \tag{4.3.2}$$

$$N_n(\mu) := |\mathfrak{N}_n(\mu)|. \tag{4.3.3}$$

For completeness we set $N_0(0) = 0$ *and* $N_n(\mu) = 0$ *if some* $\mu_i > n$.

We can now state and prove:

Lemma 4.3.7. *We have for all* $n \in \mathbb{N}$ *and* $\mu \in \mathbb{N}^s$

$$\mathbb{E}\left[e_\mu^{(n)}\right] = N_n(\mu). \tag{4.3.4}$$

Proof. The proof is similar to the proof of proposition 4.1.0.1. One has to replace $V_{k,l}$ and $\mathrm{SeT}_{k,l}$ by V_μ and SeT_μ. The only difficult point is

$$T(v_I) \neq 0 \Leftrightarrow |I^\epsilon| \le 1 \text{ for all odd } \epsilon \in \{1,c\}^s.$$

Choose any $I = (I_1,\ldots,I_s) \in \mathrm{SeT}_\mu$. As before we have to calculate

$$\sum_{\substack{g\in S_n \\ gv_I = \pm v_I}} gv_I.$$

Choose any $g = (\delta_{i,\sigma(j)}) \in S_n$ with $gv_I = \pm v_I$. Then $\sigma(I_i) = I_i$ and $\sigma(I_i^c) = I_i^c$ for $1 \le i \le s$. It follows that $\sigma(I^\epsilon) = I^\epsilon$ for all $\epsilon \in \{1,c\}^s$.
Choose $\epsilon \in \{1,c\}^s$ fixed. W.l.o.g. σ only permutes the set I^ϵ and leaves the rest fixed. It is easy to see that

$$gv_{I_i} = \left\{ \begin{array}{ll} \epsilon(\sigma) v_{I_i}, & I^\epsilon \subset I_i; \\ v_{I_i}, & I^\epsilon \not\subset I_i. \end{array} \right.$$

Now $\#\{1 \le i \le s; I^\epsilon \subset I_i\}$ is odd exactly when ϵ is odd.
We get

$$g(v_I) = \begin{cases} \epsilon(\sigma) v_I, & \epsilon \text{ odd;} \\ v_I, & \epsilon \text{ even.} \end{cases}$$

The rest follows as in lemma 4.1.6. □

Corollary 4.3.7.1. *The following hold:*

- *If* $\mu_1 > 1 + \sum_{k=2}^s \mu_k$ *then* $\mathbb{E}[e_\mu] = 0$.
- *If* $\mu_1 = 1 + \sum_{k=2}^s \mu_k$, *then* $\mathbb{E}[e_\mu] = 1$.
- *The map* $n \to \mathbb{E}\left[e_\mu^{(n)}\right]$ *is increasing and becomes stationary for* $n \ge |\mu|$.

Proof. Choose any $I \in \mathrm{SeT}_\mu$ and $\epsilon = (1, c, c, \cdots, c)$ fixed. If $I \in \mathfrak{N}_n(\mu)$ then I has to fulfill

$$|I^\epsilon| = |I_1 \setminus (I_2 \cup \cdots \cup I_s)| \le 1$$

The cardinality of the set $I_1 \setminus (I_2 \cup \cdots \cup I_s)$ is at least $\mu_1 - (\mu_2 + \cdots + \mu_s)$.
For the first part, this is ≥ 2 and we can not fulfill the above condition.
For the second part, there is only one possibility to fulfill the condition: $I_i \subset I_1$ and all I_i disjoint for $i > 1$.
Since the odd intersections only take care of the point inside the I_j's, one can add a point outside without changing the odd intersections. Therefore the function is increasing.
Because $|I_i| = \mu_i$, at most $|\mu|$ points are involved. We can assume that all points used are in $\{1, \cdots, |\mu|\}$. □

We now try to compute $N_n(\mu)$.
If $I \sim J$, we must have $|I^\epsilon| = |J^\epsilon|$ for all $\epsilon \in \{1, c\}^s$. On the other hand we know that the sets I^ϵ are all disjoint. The same is true for J^ϵ. If we have $|I^\epsilon| = |J^\epsilon|$ for all $\epsilon \in \{1, c\}^s$, we can define a permutation $\sigma \in S_n$ with $\sigma(I^\epsilon) = J^\epsilon$. It follows immediately that $\sigma(I_i) = \sigma(J_i)$ for $1 \le i \le s$, since $I_i = \bigcup_{\substack{\epsilon \in \{1,c\}^s \\ \epsilon_i = 1}} I^\epsilon$. It follows that $I \sim J$ is equivalent to $|I^\epsilon| = |J^\epsilon|$ for all $\epsilon \in \{1, c\}^s$.
Now let $I \in \mathfrak{N}_n(\mu)$ be arbitrary and set $n_\epsilon := |I^\epsilon|$. The finite sequences $(n_\epsilon)_{\epsilon \in \{1,c\}^s}$ fulfills

$$\begin{aligned} n_\epsilon &\in \mathbb{N}, && \epsilon \in \{1, c\}^s && (4.3.5a) \\ n_\epsilon &\le 1 && \text{for } \epsilon \text{ odd} && (4.3.5b) \\ \mu_i &= \sum_{\substack{\epsilon \in \{1,c\}^s \\ \epsilon_i = 1}} n_\epsilon && 1 \le i \le s && (4.3.5c) \\ n &= \sum_{\epsilon \in \{1,c\}^s} n_\epsilon. && && (4.3.5d) \end{aligned}$$

This follows from the definition of $\mathfrak{N}_n(\mu)$ (see (4.3.2)).
If $(n_\epsilon)_{\epsilon \in \{1,c\}}$ is a solution of (4.3.5) then it is easy to construct an $I \in \mathfrak{N}_n(\mu)$ with $|I^\epsilon| = n_\epsilon$.
We obtain

Lemma 4.3.8. *We have for all* $n \in \mathbb{N}$ *and all* $\mu \in \mathbb{N}^s$

$$\mathbb{E}\left[e_\mu^{(n)}\right] = N_n(\mu) = \#\textit{Solutions of } (4.3.5). \tag{4.3.6}$$

Proof. The first equality is just lemma 4.3.7. The second follows from the above considerations and the fact that an element of $I \in \mathfrak{N}_n(\mu)$ is uniquely determinate by the cardinalities of each I^ϵ. $\square$

Unfortunately we cannot give an explicit expression for $N_n(\mu)$ since it is still unkown how to solve a system like (4.3.5) in general.

4.4 Generating function for $\mathbb{E}\left[Z_n^s(x)\right]$

Even though we cannot calculate $\mathbb{E}\left[e_\mu\right]$ explicitly, we can use it to write down a generating function for $\mathbb{E}\left[Z_n^s(x)\right]$. As before let $s \in \mathbb{N}$ be fixed. We have found in (4.0.3) that

$$\mathbb{E}\left[\prod_{k=1}^{s} Z_n(\mathrm{x}_k)\right] = \sum_{\mu\in\mathbb{N}^s} \mathbb{E}\left[e_\mu^{(n)}\right](-1)^{|\mu|}\mathbf{x}^\mu$$

We know from section 4.3 that $\mathbb{E}\left[e_\mu^{(n)}\right]$ is the number of solutions of (4.3.5). The idea is to use theorem 2.6.9 to write down the generating function. We have to define a set $\mathcal{N}$ and a function $wt : \mathcal{N} \to \mathbb{C}[[\mathbf{x}, t]]$ to use theorem 2.6.9. After taking a look at (4.3.5), we set

$$\mathcal{N} := \bigoplus_{\epsilon\in\{1,c\}^s} \mathcal{N}_\epsilon \text{ with } \mathcal{N}_\epsilon := \begin{cases} \{0,1\}, & \epsilon \text{ odd;} \\ \mathbb{N}, & \epsilon \text{ even;} \end{cases} \tag{4.4.1}$$

An element of $\mathcal{N}$ can be written as $(n_\epsilon)_{\epsilon\in\{1,c\}^s}$. For shortness, we just write (n_ϵ).
If a $(n_\epsilon) \in \mathcal{N}$ is given, then there exists a unique $\mu \in \mathbb{N}^s$ and a unique $n \in \mathbb{N}$ such that (n_ϵ) is a solution of (4.3.5) for this μ and n. This follows directly from (4.3.5c) and (4.3.5d). The idea is now to define wt such that each (n_ϵ) is mapped to $\mathbf{x}^\mu t^n$. We reach this target by setting

$$wt\Big((n_\epsilon)\Big) := \mathrm{x}_1^{\sum_{\epsilon_1=1} n_\epsilon}\mathrm{x}_2^{\sum_{\epsilon_2=1} n_\epsilon}\cdots\mathrm{x}_s^{\sum_{\epsilon_s=1} n_\epsilon} t^{\sum_\epsilon n_\epsilon} \tag{4.4.2}$$

with

$$\sum_{\epsilon_1=1} := \sum_{\substack{\epsilon\in\{1,c\}^s \\ \epsilon_1=1}} \text{ and } \sum_{\epsilon} := \sum_{\epsilon\in\{1,c\}^s}$$

We set

$$f^{(s)}(\mathbf{x}, t) := \sum_{(n_\epsilon)\in\mathcal{N}} wt\Big((n_\epsilon)\Big) \tag{4.4.3}$$

and get with lemma 4.3.8 and the definition of wt

$$f^{(s)}(\mathbf{x}, t) = \sum_{n\in\mathbb{N}}\sum_{\mu\in\mathbb{N}^s}\left(\mathbb{E}\left[e_\mu^{(n)}\right]\right)\mathbf{x}^\mu t^n = \sum_{n\in\mathbb{N}}\left(\sum_{\mu\in\mathbb{N}^s}\left(\mathbb{E}\left[e_\mu^{(n)}\right]\right)\mathbf{x}^\mu\right)t^n. \tag{4.4.4}$$

We now use (4.0.3) and get

$$f^{(s)}(\mathbf{x}, t) = \sum_{n\in\mathbb{N}}\left(\mathbb{E}\left[\prod_{k=1}^{s} Z_n(-\mathrm{x}_k)\right]\right)t^n. \tag{4.4.5}$$

Remark: Why do we have $-x_k$ and not x_k in (4.4.4)? The reason is the definition of wt and $Z_n(x)$. We have

$$Z_n(x) = \sum_{k=0}^{n}(-1)^k e_k(g)x^k \text{ and } Z_n(-x) = \sum_{k=0}^{n} e_k(g)x^k$$

The minus sign in (4.4.5) would disappear if we would use $\det(I + xg)$ as the definition of $Z_n(x)$. But the final results in theorem 4.4.1 would be more complicated.
We could define the function wt in a different way too. In this case the expression for wt would be (much) more difficult and the step from (4.4.3) to (4.4.4) would be more difficult. Since the step form (4.4.3) to (4.4.4) is the most important and the most difficult, we keep it as easy as possible.

We now use theorem 2.6.9 to reformulate (4.4.3).
Since the function wt defined in (4.4.2) fulfils the conditions of the theorem 2.6.9, we only have to look at $\mathcal{N}_\epsilon$. We define $\mathbf{x}^\epsilon := \mathrm{x}_1^{\epsilon_1} \cdots \mathrm{x}_s^{\epsilon_s}$ with $\mathrm{x}_i^c := \mathrm{x}_i^0$. We have

$$f_{\mathcal{N}_\epsilon} = 1 + \mathbf{x}^\epsilon t \qquad \text{if } \epsilon \text{ is odd}$$

$$f_{\mathcal{N}_\epsilon} = 1 + \mathbf{x}^\epsilon t + (\mathbf{x}^\epsilon t)^2 + \cdots = \frac{1}{1 - \mathbf{x}^\epsilon t} \qquad \text{if } \epsilon \text{ is even.}$$

We define $\deg(\epsilon) := \#\{1 \le i \le s; \epsilon_i = 1\}$ and get

$$f^{(s)}(\mathbf{x}, t) = \prod_{\epsilon \in \{1,c\}^s} \left(1 - (-1)^{\deg(\epsilon)} \mathbf{x}^\epsilon t\right)^{(-1)^{\deg(\epsilon)+1}}. \tag{4.4.6}$$

Theorem 4.4.1. *Let $s \in \mathbb{N}$ and $x \in \mathbb{C}$. We set*

$$f^{(s)}(x, t) := \prod_{k=0}^{s} \left(1 - x^k t\right)^{\binom{s}{k}(-1)^{k+1}}. \tag{4.4.7}$$

Then

$$\mathbb{E}\left[Z_n^s(x)\right] = \left[f^{(s)}(x, t)\right]_n. \tag{4.4.8}$$

Recall: $\left[\sum_{m=1} h_m t^m\right]_n = h_n$.

Proof. The expressions of $f^{(s)}(\mathbf{x}, t)$ in (4.4.5) and (4.4.6) agree also as holomorphic functions if $\max_{1 \le i \le s} |t\mathrm{x}_i| < 1$. This is because the expression in (4.4.6) is holomorphic near 0 and theorem 2.6.9 is just a reordering.
We put $\mathrm{x}_i = -x$ for $1 \le i \le s$. We have $\mathrm{x}^\epsilon = x^{\deg(\epsilon)}$ and the number of $\epsilon \in \{1, c\}^s$ with $\mathrm{x}^\epsilon = x^k$ is $\binom{s}{k}$. The theorem now follows from (4.4.5) □

One of the reasons why we have used $\mathrm{x} = (\mathrm{x}_1, \cdots, \mathrm{x}_s)$ and not x is that we can now directly write down a generating function for $\mathbb{E}\left[\prod_{k=1}^d Z_n^{s_k}(x_k)\right]$. We state here only the case $d = 2$ since there are only minor differences between $d = 2$ and $d > 2$, but the expressions for $d = 2$ are much simpler than for $d > 2$.

Theorem 4.4.2. *Let $s_1, s_2 \in \mathbb{N}$ and $x_1, x_2 \in \mathbb{C}$. We set*

$$f^{(s_1,s_2)}(x_1, x_2, t) := \prod_{k_1=0}^{s_1} \prod_{k_2=0}^{s_2} \left(1 - x_1^{k_1} x_2^{k_2} t\right)^{\binom{s_1}{k_1}\binom{s_2}{k_2}(-1)^{k_1+k_2+1}}. \tag{4.4.9}$$

Then

$$\mathbb{E}\left[Z_n^{s_1}(x_1) Z_n^{s_2}(x_2)\right] = \left[f^{(s_1,s_2)}(x_1, x_2, t)\right]_n.$$

Proof. We put $s = s_1 + s_2$, $\mathrm{x}_i = -x_1$ for $1 \le i \le s_1$ and $\mathrm{x}_i = -x_2$ for $s_1 + 1 \le i \le s_1 + s_2$. Each factor that can appear in (4.4.6) has now the form $\left(1 - x_1^{k_1} x_2^{k_2} t\right)^{(-1)^{k_1+k_2+1}}$. The rest is just counting. □

4.5 Convergence for $|x| < 1$

We have found the generating function of $\mathbb{E}\left[Z_n^s(x)\right]$ and now use it to calculate the asymptotic behavior inside the unit disc. We have

Theorem 4.5.1. *Choose $|x_1| < 1, |x_2| < 1$ fixed. We then have for $s_1, s_2 \in \mathbb{N}$*

$$\lim_{n\to\infty} \mathbb{E}\left[Z_n^{s_1}(x_1) Z_n^{s_2}(x_2)\right] = \left(\prod_{k_1=0}^{s_1}\prod_{k_2=0}^{s_2}\right)' \left(1 - x_1^{k_1} x_2^{k_2}\right)^{\binom{s_1}{k_1}\binom{s_2}{k_2}(-1)^{k_1+k_2+1}}. \tag{4.5.1}$$

where the prime indicates that the factor $k_1 = k_2 = 0$ is omitted.

Proof. First proof:
This proof is based on theorem 2.6.6. We therefore have to check that the assumptions of theorem 2.6.6 are fulfilled. It follows immediately from (4.4.9) and $|x_1| < 1, |x_2| < 1$ that

$$f^{(s_1,s_2)}(x_1, x_2, t) = \frac{1}{1-t} F(t) \tag{4.5.2}$$

with $F(t)$ holomorphic on $B_{1+\delta}(0) = \{x \in \mathbb{C}; |x| < 1 + \delta\}$ for some $\delta > 0$ with

$$F(1) = \left(\prod_{k_1=0}^{s_1}\prod_{k_2=0}^{s_2}\right)' \left(1 - x_1^{k_1} x_2^{k_2}\right)^{\binom{s_1}{k_1}\binom{s_2}{k_2}(-1)^{k_1+k_2+1}}. \tag{4.5.3}$$

Therefore $f^{(s_1,s_2)}(x_1, x_2, t)$ is holomorphic on some Δ_0 (see definition 2.6.5) and

$$(1-t) f^{(s_1,s_2)}(x_1, x_2, t) = F(1) + O(1-t) \text{ for } t \to 1.$$

We now can apply theorem 2.6.6 and are done.

Second proof:
We give here a direct proof of theorem 4.5.1. We formulate this as a lemma

Lemma 4.5.2. *Let $a, b \in \mathbb{N}$ and $y_1, \cdots, y_a, z_1, \cdots, z_b$ be complex numbers with $\max\{|y_i|, |z_i|\} < 1$. Then*

$$\lim_{n\to\infty} \left[\frac{1}{1-t} \frac{\prod_{i=1}^a (1 - y_i t)}{\prod_{i=1}^b (1 - z_i t)}\right]_n = \frac{\prod_{i=1}^a (1 - y_i)}{\prod_{i=1}^b (1 - z_i)}. \tag{4.5.4}$$

It is obvious that theorem 4.5.1 follows from lemma 4.5.2.

Proof of lemma 4.5.2. We show this by induction on the number of factors.
For $a = b = 0$ there is nothing to do, since $\left[\frac{1}{1-t}\right]_n = 1$.
Induction $(a, b) \to (a + 1, b)$.
We set

$$g(t) := \frac{1}{1-t} \frac{\prod_{i=1}^a (1 - y_i t)}{\prod_{i=1}^b (1 - z_i t)}, \quad \gamma := \frac{\prod_{i=1}^a (1 - y_i)}{\prod_{i=1}^b (1 - z_i)}.$$

We know by induction that $\lim_{n\to\infty}[g(t)]_n = \gamma$. We get

$$\begin{aligned}\lim_{n\to\infty} \left[\frac{1}{1-t} \frac{\prod_{i=1}^{a+1} (1 - y_i t)}{\prod_{i=1}^b (1 - z_i t)}\right]_n &= \lim_{n\to\infty} \left[g(t)(1 - y_{a+1} t)\right]_n = \lim_{n\to\infty} \left[g(t)\right]_n - \lim_{n\to\infty} y_{a+1} \left[g(t)\right]_{n-1} \\ &= (1 - y_{a+1})\gamma.\end{aligned}$$

Induction $(a, b) \to (a, b + 1)$.
This case is slightly more difficult. We define $g(t)$ and γ as above and write $z = z_{b+1}$. We get

$$\left[\frac{1}{1-t}\frac{\prod_{i=1}^{a}(1-y_i t)}{\prod_{i=1}^{b+1}(1-z_i t)}\right]_n = \left[\frac{g(t)}{(1-zt)}\right]_n = \left[g(t)\sum_{k=0}^{\infty}(zt)^k\right]_n = \sum_{k=0}^{n} z^k \left[g(t)\right]_{n-k}.$$

Let $\epsilon > 0$ be arbitrary. Since $[g(t)]_n \to \gamma$, we know that there exists a $n_0 \in \mathbb{N}$ with $|[g(t)]_n - \gamma| < \epsilon$ for all $n \geq n_0$. We have

$$\sum_{k=0}^{n} z^k \left[g(t)\right]_{n-k} = \sum_{k=0}^{n-n_0} z^k \left[g(t)\right]_{n-k} + \sum_{k=0}^{n_0-1} z^{n-k} \left[g(t)\right]_k.$$

Since $|z| < 1$, the second sum converges to 0 as $n \to \infty$. But

$$\left|\sum_{k=0}^{n-n_0} \gamma z^k - \sum_{k=0}^{n-n_0} \left[g(t)\right]_{n-k} z^k\right| \leq |\epsilon| \sum_{k=0}^{n-n_0} \left|z^{n-k}\right| \leq \frac{\epsilon}{1-|z|}.$$

Since ϵ was arbitrary, we are done. □

□

Corollary 4.5.2.1. *For each $s \in \mathbb{N}$ and $|x| < 1$, we have*

$$\lim_{n\to\infty} \mathbb{E}\left[|Z_n(x)|^{2s}\right] = \prod_{k=1}^{s}(1-|x|^{2k})^{-\binom{s}{k}^2} \left|\prod_{k_1>k_2}(1-x^{k_1}\overline{x}^{k_2})^{\binom{s}{k_1}\binom{s}{k_2}(-1)^{k_1+k_2+1}}\right|^2. \quad (4.5.5)$$

Proof. We know from proposition 4.1.0.1, that $\overline{Z_n(x)} = Z_n(\overline{x})$. We put $s_1 = s_2 = s$, $x_1 = x$ and $x_2 = \overline{x}$ and apply theorem 4.5.1 □

Remark: The corollary is in fact true for all $s \in \mathbb{R}$. This follows with the same argument from theorem 4.7.4 (see section 4.7).

4.6 Growth Rates for $|x| = 1$

We consider only the two variable case with $x = x_1 = \overline{x_2}$. We assume $s_1, s_2 \in \mathbb{N}$, $|x| = 1$ and x not a root of unity, i.e $x^k \neq 1$ for all $k \in \mathbb{Z} \setminus \{0\}$.
We first calculate the growth rate of $\mathbb{E}\left[Z_n^{s_1}(x)Z_n^{s_2}(\overline{x})\right]$ for $s_2 = 0$ (see lemma 4.6.2 and 4.6.3) and then for s_2 arbitrary (see theorem 4.6.5).
The main results in this section can be obtained by using theorem 2.6.6. We do not use it because we can use here a more direct and simpler argument. This argument is a partial fraction decomposition. The advantage is that we can calculate directly the asymptotic behavior with a simple two line calculation.

4.6.1 Case $s_2 = 0$

We use the generating function of theorem 4.4.1. We have to calculate the growth rate of $\left[f^{(s)}(x,t)\right]_n$. But $f^{(s)}(x,t)$ is a quite complicated expression and we therefore express it in a different way. Since $f^{(s)}(x,t)$ is a rational function, we can do a partial fractional decomposition with respect to t (and x fixed).

$$f^{(s)}(x,t) = \prod_{k=0}^{s} \left(1 - x^k t\right)^{(-1)^{k+1}\binom{s}{k}} = P(t) + \sum_{k \text{ even}} \sum_{l=1}^{\binom{s}{k}} \frac{a_{k,l}}{(1 - x^k t)^l} \tag{4.6.1}$$

where P is a polynomial and $a_{k,l}$ are complex constants. Note that at this point we need the condition that x is not a root of unity. If x is a root of unity, some factors can be equal and cancel or increase the power. For example, we have $f^{(s)}(1,t) = 1$ for all $s \in \mathbb{C}$.

What is the growth rate of $\left[\frac{1}{(1-x^k t)^l}\right]_n$? We have for $l \in \mathbb{N}$ and $|t| < 1$

$$\frac{1}{(1-t)^l} = \frac{1}{(l-1)!} \sum_{n=0}^{\infty} \Big(\prod_{k=1}^{l-1} (n+k) \Big) t^n. \tag{4.6.2}$$

This equation can be shown by differentiating the geometric series. We get

$$\left[\frac{1}{(1-x^k t)^l}\right]_n \sim \frac{n^{l-1} (x^k)^n}{(l-1)!}.$$

Recall that $A(n) \sim B(n)$ if $\lim_{n\to\infty} \frac{A(n)}{B(n)} = 1$. Since $|x| = 1$, we have

$$\left| \left[\frac{1}{(1-x^k t)^l}\right]_n \right| \sim \frac{n^{l-1}}{(l-1)!}. \tag{4.6.3}$$

We know from (4.6.3) the growth rate of each summand in (4.6.1). Since we have only finitely many summands, only these $\frac{a_{k,l}}{(1-x^k t)^l}$ are relevant with l maximal and $a_{k,l} \neq 0$. We also know $a_{k,\binom{s}{k}} \neq 0$, since the pole-order in $\overline{x}^k$ on both sides of (4.6.1) has to be equal. Before we can write down the growth rate of $\mathbb{E}\left[Z_n^s(x)\right]$, we have to define

Definition 4.6.1. *Let $s, k_0 \in \mathbb{N}$ with $0 \leq k_0 \leq s$ and k_0 even. We set*

$$C(k_0) := \frac{1}{\left(\binom{s}{\lfloor s/2 \rfloor} - 1\right)!} \prod_{k \neq k_0} \left(1 - x^k \overline{x}^{k_0}\right)^{(-1)^{k+1}\binom{s}{k}}. \tag{4.6.4}$$

We put everything together and get

Lemma 4.6.2. *We have for $s \neq 4m+2$*

$$\mathbb{E}\left[Z_n^s(x)\right] \sim n^{\binom{s}{k_0}-1} C(k_0) (x^{k_0})^n \tag{4.6.5}$$

with

$$k_0 := \begin{cases} 2m, & \text{for } s = 4m, \\ 2m, & \text{for } s = 4m+1, \\ 2m+2, & \text{for } s = 4m+3. \end{cases}$$

Proof. We have to calculate $M := \max_{k \text{ even}} \binom{s}{k}$. A straight forward verification shows that $M = \binom{s}{k_0} = \binom{s}{\lfloor s/2 \rfloor}$ and that there is only one summand with exponent M in the case $s \neq 4m+2$. We apply (4.6.3) and get

$$\mathbb{E}\left[Z_n^s(x)\right] \sim n^{\binom{s}{k_0}-1} \frac{a_{k_0,\binom{s}{k_0}}}{\left(\binom{s}{k_0} - 1\right)!} (x^{k_0})^n.$$

It follows with residue calculus that $a_{k_0,\binom{s}{k_0}} = \prod_{k \neq k_0} \left(1 - x^k \overline{x}^{k_0}\right)^{(-1)^{k+1}\binom{s}{k}}$. This proves (4.6.5).

∎

We see in the next lemma that there can appear more than one constant. This is the reason why we write $C(k_0)$ for the constant and not C or $C(s,x)$.
The case $s = 4m+2$ is a little bit more difficult, since there are two maximal terms, i.e. $\binom{4m+2}{2m} = \binom{4m+2}{2m+2}$.

Lemma 4.6.3. *If $s = 4m+2$ then*

$$\mathbb{E}\left[Z_n^s(x)\right] \sim n^{\binom{4m+2}{2m}-1}\Big(C(2m)(x^{2m})^n + C(2m+2)(x^{2m+2})^n\Big) \tag{4.6.6}$$

with $C(2m)(x^{2m})^n + C(2m+2)(x^{2m+2})^n = 0$ for at most one n.

Proof. A straightforward verification as in lemma 4.6.2 shows that $M = \binom{4m+2}{2m} = \binom{4m+2}{2m+2}$.
Now we have two summands with a maximal l.
To prove (4.6.6), we have to that show $C(2m)(x^{2m})^n + C(2m+2)(x^{2m+2})^n = 0$ for only finitely many n. But $C(2m)(x^{2m})^n + C(2m+2)(x^{2m+2})^n = 0$ implies $x^{2n} = -\frac{C(2m)}{C(2m+2)}$.
Since x is not a root of unity, all x^k are different. □

4.6.2 Case with s_2 arbitrary

We argue as before.
Some factors appearing in $f^{(s_1,s_2)}(x,\overline{x},t)$ (see (4.4.7)) are equal, so we have to collect them before we can write down the partial fraction decomposition.

$$f^{(s_1,s_2)}(x,\overline{x},t) = \prod_{k_1=0}^{s_1}\prod_{k_2=0}^{s_2}\left(1-x^{k_1}\overline{x}^{k_2}t\right)^{(-1)^{k_1+k_2+1}\binom{s_1}{k_1}\binom{s_2}{k_2}} = \prod_{k=-s_2}^{s_1}(1-x^k t)^{S(k)} \tag{4.6.7}$$

with

$$S(k) = \sum_{j=0}^{\infty}\binom{s_1}{k+j}\binom{s_2}{j}(-1)^{k+2j+1}.$$

To calculate $S(k)$ explicitly, we need Vandermonde's identity for binomial coefficients:

$$\binom{m_1+m_2}{q} = \sum_{j=0}^{\infty}\binom{m_1}{q-j}\binom{m_2}{j}. \tag{4.6.8}$$

We get with $m_1 = s_1, m_2 = s_2, q = s_1 - k$ and $\binom{m}{k} = \binom{m}{m-k}$ that

$$S(k) = (-1)^{k+1}\binom{s_1+s_2}{s_1-k} \tag{4.6.9}$$

and therefore

$$f^{(s_1,s_2)}(x,\overline{x},t) = \prod_{k=-s_2}^{s_1}(1-x^k t)^{(-1)^{k+1}\binom{s_1+s_2}{s_1-k}} = \prod_{k=-s_2}^{s_1}(1-x^k t)^{(-1)^{k+1}\binom{s_1+s_2}{s_2+k}}.$$

Before we look at the growth rate of $\mathbb{E}\left[Z_n^{s_1}(x)Z_n^{s_2}(\overline{x})\right]$, we define

Definition 4.6.4. *We set for $s_1, s_2 \in \mathbb{N}, k_0 \in \mathbb{Z}$ with $-s_2 \le k_0 \le s_1$ and k_0 even*

$$C(k_0) = C(s_1,s_2,k_0,x) = \frac{1}{\left(\binom{s_1+s_2}{s_2+k_0}-1\right)!}\prod_{k\neq k_0}(1-x^k\overline{x}^{k_0})^{(-1)^{k+1}\binom{s_1+s_2}{s_2+k}}. \tag{4.6.10}$$

Remark: definition 4.6.1 is a special case of definition 4.6.4. Therefore there is no danger of confusion and we can write $C(k_0)$ for both of them.

Theorem 4.6.5.

- *If* $s_1 - s_2 \neq 4m+2$ *then*

$$\mathbb{E}\left[Z_n^{s_1}(x) Z_n^{s_2}(\overline{x})\right] \sim n^{\binom{s_1+s_2}{\lfloor (s_1+s_2)/2 \rfloor}-1} C(k_0) (x^{k_0})^n \tag{4.6.11}$$

with $k_0 := \begin{cases} \frac{s_1-s_2}{2}, & \text{for } s_1 - s_2 = 4m, \\ \frac{s_1-s_2-1}{2}, & \text{for } s_1 - s_2 = 4m+1, \\ \frac{s_1-s_2+1}{2}, & \text{for } s_1 - s_2 = 4m+3. \end{cases}$

- *If* $s_1 - s_2 = 4m+2$ *we set* $k_0 := \frac{s_1-s_2}{2}$. *Then*

$$\mathbb{E}\left[Z_n^{s_1}(x) Z_n^{s_2}(\overline{x})\right] \sim n^{\binom{s_1+s_2}{k_0-1}} \Big(C(k_0-1)(x^{k_0-1})^n + C(k_0+1)(x^{k_0+1})^n \Big). \tag{4.6.12}$$

Additionally, for every even k_0 *with* $-s_2 \leq k_0 \leq s_1$

$$C(s_1, s_2, -k_0) = \overline{C(s_2, s_1, k_0)}.$$

Proof. We prove here only the case $s_1 + s_2 = 4p+1$ and s_2 even or odd. The other cases are similar.
We have to calculate

$$M := \max_{k \text{ even}} \binom{s_1+s_2}{s_2+k}.$$

We know that $\max \binom{4p+1}{k} = \binom{4p+1}{2p} = \binom{4p+1}{2p+1}$. If s_2 is even then $k+s_2$ runs through all even numbers between 0 and s_1+s_2. Therefore $M = \binom{4p+1}{2p}$ and the maximum is attained for $k_0 = 2p - s_2 = \frac{s_1+s_2-1}{2} - s_2 = \frac{s_1-s_2-1}{2}$. We have in this case $s_1 - s_2 = 4p+1-2s_2 = 4m+1$ and formula (4.6.11) follows from (4.6.3). The argument for s_2 odd is similar.
It remains to show that

$$C(s_1, s_2, -k_0) = C(s_2, s_1, k_0).$$

This follows from

$$\begin{aligned} C(s_1, s_2, -k_0, x) &= \frac{1}{\left(\binom{s_1+s_2}{s_2-k_0} - 1\right)!} \prod_{\substack{k=-s_2 \\ k \neq -k_0}}^{s_1} \left(1 - x^k \overline{x}^{-k_0}\right)^{(-1)^{k+1}\binom{s_1+s_2}{s_2+k}} \\ &= \frac{1}{\left(\binom{s_1+s_2}{s_1+k_0} - 1\right)!} \prod_{\substack{k=-s_1 \\ k \neq k_0}}^{s_2} \left(1 - x^{-k} x^{k_0}\right)^{(-1)^{k+1}\binom{s_1+s_2}{s_1+k}} \\ &= \overline{C(s_2, s_1, k_0, x)} \end{aligned}$$

□

It follows

Corollary 4.6.5.1.

$$\mathbb{E}\left[|Z_n(x)|^{2s}\right] \sim n^{\binom{2s}{s}-1} \frac{\prod_{k=1}^{s} |1-x|^{2\binom{2s}{s+k}}}{\left(\binom{2s}{s} - 1\right)!}.$$

Proof. Put $s_1 = s_2 = s$ in theorem 4.6.5. $\square$

Corollary 4.6.5.2.

$$\mathrm{Var}\Big(Z_n(x)\Big) \sim n\ |1-x|^2.$$

Proof. We have $\mathbb{E}\left[|Z_n(x)|^2\right] \sim C(1,1,0,x)n$ and $\mathbb{E}\left[Z_n(x)\right] = 1-x$ (see (4.1.3)). $\square$

4.6.3 The real and the imaginary parts

We mentioned in the introduction the results in [17]. Do we have the same results for $Z_n(x)$?
We first look at the expectation and the variance of the real and the imaginary of $Z_n(x)$. We set $R_n(x) := \mathrm{Re}\Big(Z_n(x)\Big)$ and $I_n(x) := \mathrm{Im}\Big(Z_n(x)\Big)$. We have

Lemma 4.6.6. *We write $x = e^{i\varphi}$. Then*

1. $\mathbb{E}\left[R_n(x)\right] = 1 - \cos(\varphi)$,
2. $\mathbb{E}\left[I_n(x)\right] = -\sin(\varphi)$,
3. $\mathrm{Var}\Big(R_n(x)\Big) \sim n\frac{|1-x|^2}{2}$,
4. $\mathrm{Var}\Big(I_n(x)\Big) \sim n\frac{|1-x|^2}{2}$,
5. $\mathrm{Corr}(R_n, I_n) \to 0$ *for* $n \to \infty$.

Proof. (1) and (2) follow from (4.1.3).
Now we prove (3) and (4). We use the growth rates for $s_1 = s_2 = 1$ and $s_1 = 2, s_2 = 0$. We only give the important constants explicitly.

$$\mathbb{E}\left[Z_n(x)Z_n(\overline{x})\right] = \mathbb{E}\left[R_n^2 + I_n^2\right] \sim |1-x|^2 n, \tag{4.6.13a}$$
$$\mathbb{E}\left[Z_n^2(x)\right] = \mathbb{E}\left[R_n^2 + 2iR_nI_n - I_n^2\right] \sim C_1 + C_2(x^2)^n. \tag{4.6.13b}$$

We calculate (4.6.13a) $\pm$ Re(4.6.13b) and get

$$\mathbb{E}\left[2R_n^2\right] \sim |1-x|^2 n$$

$$\mathbb{E}\left[2I_n^2\right] \sim |1-x|^2 n.$$

We now prove the last point. We know from (4.6.13b) that

$$\mathrm{Cov}(R_n, I_n) = \mathbb{E}\left[R_nI_n\right] + \sin(\varphi)(1-\cos(\varphi)) \sim C_4 + C_5\sin(2n\varphi) + C_6\cos(2n\varphi).$$

The point (5) now follows from (3) and (4). $\square$

What are the growth rates or $\mathbb{E}\left[R_n^s\right]$ and $\mathbb{E}\left[I_n^s\right]$? We need the following lemma to answer this question.

Lemma 4.6.7. *Let $s \in \mathbb{N}$ and $z = x + iy$ be given with $x, y \in \mathbb{R}$. Then*

$$x^s = \frac{1}{2^s}\sum_{k=0}^{s}\binom{s}{k}z^k\overline{z}^{s-k}; \qquad y^s = \frac{1}{(2i)^s}\sum_{k=0}^{s}(-1)^{s+k}\binom{s}{k}z^k\overline{z}^{s-k} \tag{4.6.14}$$

Proof. We argue with induction. If $s=1$ then $x=\frac{1}{2}(z+\overline{z})$.
$s \to s+1$: We have

$$x^{s+1} = x\, x^s = \frac{1}{2}(z+\overline{z})\frac{1}{2^s}\sum_{k=0}^{s}\binom{s}{k} z^k \overline{z}^{s-k} = \frac{1}{2^{s+1}}\sum_{k=0}^{s+1}\binom{s+1}{k} z^k \overline{z}^{(s+1)-k}.$$

The proof for y^s is similar. □

We then have

Theorem 4.6.8. *Choose any $s \in \mathbb{N}$ and write $x = e^{i\varphi}$ with $\varphi \in [0,2\pi] \setminus 2\pi\mathbb{Q}$. Then there exist (real) constants $a_{2k} = a_{2k}(\varphi,s), b_{2k} = b_{2k}(\varphi,s), c_{2k} = c_{2k}(\varphi,s)$ and $d_{2k} = d_{2k}(\varphi,s)$ for $0 \le k \le \lfloor (s+1)/4 \rfloor$ with*

$$\mathbb{E}\left[R_n^s\right] \sim n^{\binom{s}{\lfloor s/2 \rfloor}} \left(\sum_{k=0}^{\lfloor (s+1)/4 \rfloor} a_{2k}\cos\big((2k)n\varphi\big) + b_{2k}\cos\big((2k)n\varphi\big) \right), \tag{4.6.15}$$

$$\mathbb{E}\left[I_n^s\right] \sim n^{\binom{s}{\lfloor s/2 \rfloor}} \left(\sum_{k=0}^{\lfloor (s+1)/4 \rfloor} c_{2k}\sin\big((2k)n\varphi\big) + d_{2k}\sin\big((2k)n\varphi\big) \right). \tag{4.6.16}$$

At least one a_{2k} or b_{2k} and one c_{2k} or $d_{2k}$is not equal to zero.

Proof. We only prove the behavior for $\mathbb{E}\left[R_n^s\right]$ and $s=4m$. The other cases are similar.
We have

$$\begin{aligned}
\mathbb{E}\left[R_n^s\right] &= \frac{1}{2^s}\sum_{k=0}^{s}\binom{s}{k}\mathbb{E}\left[Z_n^k(x)Z_n^{s-k}(\overline{x})\right] \\
&= \frac{1}{2^s}\left(\sum_{\substack{k=0 \\ k \text{ even}}}^{s}\binom{s}{k}\mathbb{E}\left[Z_n^k(x)Z_n^{s-k}(\overline{x})\right] + \sum_{\substack{k=0 \\ k \text{ odd}}}^{s}\binom{s}{k}\mathbb{E}\left[Z_n^k(x)Z_n^{s-k}(\overline{x})\right] \right).
\end{aligned}$$

We now apply theorem 4.6.5. If k is odd then $k-(4m-k) = 4p+2$ for a $p \in \mathbb{Z}$ and the growth rate of $\mathbb{E}\left[Z_n^k(x)Z_n^{s-k}(\overline{x})\right]$ is $\binom{4m}{2m+1}(\dots)$. If k is even then $k-(4m-k)=4p$ for a $p \in \mathbb{Z}$ and the growth rate of $\mathbb{E}\left[Z_n^k(x)Z_n^{s-k}(\overline{x})\right]$ is $\binom{4m}{2m}(\dots)$. It is therefore sufficient to look at even k. We get

$$\begin{aligned}
\mathbb{E}\left[R_n^s\right] &\sim n^{\binom{4m}{2m}-1} \left(\frac{1}{2^s}\sum_{\substack{k=0 \\ k \text{ even}}}^{s}\binom{s}{k}\left(x^{\frac{k-(s-k)}{2}}\right)^n C\left(\frac{k-(s-k)}{2}\right)\right) \\
&\sim n^{\binom{4m}{2m}-1} \left(\frac{1}{2^s}\sum_{k=-m}^{m}\binom{s}{2m+2k}(x^{2k})^n C(2k)\right) \\
&\sim n^{\binom{4m}{2m}-1}\frac{1}{2^s} \left(\binom{s}{2m}C(0) + 2\sum_{k=1}^{m}\binom{s}{2m+2k}\Big(\mathrm{Re}(C(2k))\cos\big((2k)n\varphi\big) \right. \\
&\qquad \left. -2\sum_{k=1}^{m}\binom{s}{2m+2k}\mathrm{Im}(C(2k))\sin\big((2k)n\varphi\big)\Big)\right).
\end{aligned}$$

We have used in the last inequality that $C(s_1,s_2,-k_0) = \overline{C(s_2,s_1,k_0)}$.
This proves (4.6.15) if we can show that the last bracket is equal zero only for finitely many

n (and x fixed).
We define

$$g(t) := \binom{s}{2m} C(0) + 2\sum_{k=1}^{m} \binom{s}{2m+2k} \Big(\mathrm{Re}(C(2k)) \cos\big((2k)t\big) - \mathrm{Im}(C(2k)) \sin\big((2k)t\big)\Big).$$

Suppose there are infinitely many (different) $n \in \mathbb{N}$ with $g(n\varphi) = 0$. All numbers $n\varphi$ are different modulo 2π, since $\varphi \notin 2\pi\mathbb{Q}$. Therefore there are infinite many $t \in [0, 2\pi]$ with $g(t) = 0$. But $g(t)$ is a non trivial linear combination of $\cos(\cdot)$ and $\sin(\cdot)$ and is therefore a holomorphic function in t. It follows immediately from the identity theorem (see [14]) that $g(t) \equiv 0$. This is a contradiction since the functions $\cos(m_1 t)$ and $\sin(m_2 t)$ are linearly independent for $m_1 \geq 0, m_2 > 0$. □

4.7 Holomorphicity in s

The generating function in theorem 4.4.2 can be viewed as a formal power series or as a convergent power series. This is possible since the product is finite. If we allow s to be complex, then infinitely many $\binom{s}{k}$ are non zero and the product in theorem 4.4.2 can be viewed as convergent power series only. We have most of the time s and x fixed, but the product is in fact a holomorphic function in (x, s) for $s \in \mathbb{C}$ and $|x| < 1$. We prove the holomorphicity since we need it in the probabilistic proof of theorem 4.7.4.

4.7.1 Extensions of the definitions

Before we can state the theorems for $s \in \mathbb{C}$, we have to extend the definitions of the functions appearing there. We assume for simplicity that $s_2 = 0$ and set $f^{(s)}(x,t) = \prod_{k=0}^{\infty} \left(1 - x^k t\right)^{\left(\binom{s}{k}(-1)^{k+1}\right)}$. We now show that $Z_n^s(x)$ and $f^{(s)}(x,t)$ can be extended to holomorphic functions in (x, s) resp. in (x, s, t). The extension of $Z_n^s(x)$ is straight forward.

Definition 4.7.1. *We set for $s \in \mathbb{C}$ and $|x| < 1$*

$$Z_n^s(x) := \prod_{m=1}^{n} (1 - x^m)^{(sC_m^{(n)})}.$$

We use the principal branch of the logarithm $\log(.)$ to define a^b with $a \notin \mathbb{R}_-$. $Z_n^s(x)$ is well defined for $|x| < 1$ since $\mathrm{Re}(1 - x^m) > 0$. It is clear that $Z_n^s(x)$ is a holomorphic function in (x, s) and agrees with the old function for $s \in \mathbb{N}$.
We now look at $\binom{s}{k}$. We set

$$\binom{s}{k} := \frac{s(s-1)\cdots(s-k+1)}{k!} = \prod_{m=1}^{k} \frac{s-m+1}{m}. \tag{4.7.1}$$

Obviously, $\binom{s}{k}$ can be extended to a holomorphic function in s. We set

$$f^{(s)}(x,t) := \prod_{k=0}^{\infty} \left(1 - x^k t\right)^{\left(\binom{s}{k}(-1)^{k+1}\right)}.$$

The factor $(1 - x^k t)^{(..)}$ is well defined, because $\mathrm{Re}(1 - x^k t) > 0$ for $|x| \leq 1$ and $|t| < 1$.
This definition agrees with the old one, since $\binom{s}{k} = 0$ for $k > s$ and $s, k \in \mathbb{N}$. Of course we have to show that the infinite product is convergent.

Lemma 4.7.2. *The function $f^{(s)}(x,t)$ is a holomorphic function in (x,s,t) for $s \in \mathbb{C}$ and $|x|<1, |t|<1$.*

Proof. Choose $K \subset \mathbb{C}$ and $0<r<1$ fix with K compact. We prove the lemma for $s \in K$ and $|x|<r$. This proves the lemma since K and r are arbitrary.
We have

$$\sup_{s\in K}\left|\frac{s-m+1}{m}\right| \to 1 \text{ for } m \to \infty.$$

We obtain that for each $a>1$ there exists a $C=C(a,K)$ with $|\binom{s}{k}| \leq Ca^k$ for all $k \in \mathbb{N}, s \in K$.
We choose an $a>1$ such that $ra<1$ and set $\log(y) := |y| + i\pi$ for $y \in \mathbb{R}_-$. Then

$$\begin{aligned}\sum_{k=1}^{\infty}\left|\log\left((1-x^kt)^{\left((-1)^{k+1}\binom{s}{k}\right)}\right)\right| &\leq \sum_{k=1}^{\infty}\left|\binom{s}{k}\log(1-x^kt)\right| \leq \sum_{k=1}^{\infty}\left|\binom{s}{k}\right|(C(r)|x^kt|)\\ &\leq C(r)\sum_{k=1}^{\infty}C(a)(ar)^k < \infty,\end{aligned}$$

where we have used the well know fact that for each $r<1$ there exists a constant $C(r)$ with $|\log(1+y)| \leq C(r)|y|$ for $|y|<r$. Since this upper bound is independent of x, we can find a k_0 such that

$$\arg\left((1-x^k)^{\left((-1)^{k+1}\binom{s}{k}\right)}\right) \in [-\pi/2, \pi/2] \; \forall k \geq k_0 \text{ and } |x|<r.$$

Therefore $\sum_{k=k_0}^{\infty}\log\left((1-x^k)^{\left((-1)^{k+1}\binom{s}{k}\right)}\right)$ is a holomorphic function. This proves the holomorphicity of $f^{(s)}(x)$. $\square$

4.7.2 Generating function for $s \in \mathbb{C}$

It is not very surprising that theorem 4.4.2 is also true for complex s. We have

Theorem 4.7.3. *Let $s_1, s_2 \in \mathbb{C}$. We then have for $x_1, x_2, t \in \mathbb{C}$ with $\max\{|x_1|,|x_2|,|t|\}<1$*

$$\mathbb{E}\left[Z_n^{s_1}(x_1)Z_n^{s_2}(x_2)\right] = \left[\prod_{k_1=0}^{\infty}\prod_{k_2=0}^{\infty}\left(1-x_1^{k_1}x_2^{k_2}t\right)^{\left(\binom{s_1}{k_1}\binom{s_2}{k_2}(-1)^{k_1+k_2+1}\right)}\right]_n. \tag{4.7.2}$$

The product in (4.7.2) is holomorphic in (t, x_1, x_2, s_1, s_2).

Proof. This theorem is a corollary of theorem 5.7.2 (see later) with $f(x_1,x_2) = (1-x_1^{s_1})(1-x_2^{s_2})$. The only thing that on needs is that $\sum_{k=0}^{\infty}\binom{s}{k}x^k$ is absolutely convergent in the cases $s \in \mathbb{C}, |x|<1$ (see [14]). $\square$

We can use this theorem to calculate the asymptotic behavior inside the unit disc. Unfortunately we cannot apply theorem 2.6.6 for $|x|=1$ since there are infinitely many singularities on S^1. The asymptotic behavior for $|x|=1$ remains as an open question.

4.7.3 Asymptotics for $|x|<1$

Theorem 4.5.1 can also be extended to $s \in \mathbb{C}$. We have

Theorem 4.7.4. *We have for $s_1, s_2 \in \mathbb{C}$ and* $\max\{|x_1|, |x_2|\} < 1$

$$\mathbb{E}\left[Z_n^{s_1}(x_1) Z_n^{s_2}(x_2)\right] \to \left(\prod_{k_1=0}^{\infty} \prod_{k_2=0}^{\infty}\right)' \left(1 - x_1^{k_1} x_2^{k_2}\right)^{\left(-\binom{s_1}{k_1}\binom{s_2}{k_2}(-1)^{k_1+k_2}\right)} \tag{4.7.3}$$

for $n \to \infty$ point-wise in x_1, x_2. The prime indicates that the factor $k_1 = k_2 = 1$ is omitted.

It follows as in lemma 4.7.2 that the product on the RHS of (4.7.3) is a holomorphic function. We have proven theorem 4.5.1 directly with a induction over the number of factors and with function theory. Since the product has now infinitely many factors, we can not use induction over the number of factors. The function theoretic proof is still true. One can in fact use the same proof word by word. We give in section 5.8 a probabilistic proof for the more general case of associated class functions, which includes the characteristic polynomial.

4.7.4 Corollaries of theorem 4.7.4

Corollary 4.7.4.1. *We have for $s_1, s_2 \in \mathbb{C}$ and* $\max\{|x_1|, |x_2|\} < 1$

$$\mathbb{E}\left[\frac{Z_n^{s_1}(x_1)}{Z_n^{s_2}(x_2)}\right] \to \prod_{\substack{k_1,k_2\in\mathbb{N}\\ k_1+k_2\neq 0}} \left(1 - x_1^{k_1} x_2^{k_2}\right)^{-\left(\binom{s_1}{k_1}\binom{s_2+k_2-1}{k_2}(-1)^{k_1}\right)} \qquad (n\to\infty).$$

Proof. We use the definition of $\binom{s}{k}$ in (4.7.1) for $s \in \mathbb{C}, k \in \mathbb{N}$ (see later).
We apply theorem 4.7.4 and the identity $\binom{-s}{k} = (-1)^k \binom{s+k-1}{k}$. □

Corollary 4.7.4.2. *We have for $x_1, x_2, x_3, x_4, s_1, s_2, s_3, s_4 \in \mathbb{C}$ with* $\max\{|x_i|\} < 1$

$$\mathbb{E}\left[\frac{Z_n^{s_1}(x_1) Z_n^{s_2}(x_2)}{Z_n^{s_3}(x_3) Z_n^{s_4}(x_4)}\right] \to \prod_{\substack{k_1,k_2,k_3,k_4\in\mathbb{N}\\ k_1+k_2+k_3+k_4\neq 0}} \left(1 - x_1^{k_1} x_2^{k_2} x_3^{k_3} x_4^{k_4}\right)^{\binom{s_1}{k_1}\binom{s_2}{k_2}\binom{s_3+k_3-1}{k_3}\binom{s_4+k_3-1}{k_4}(-1)^{k_1+k_2+1}}$$

We can also calculate the limit of the Mellin-Fourier-transformation of $Z_n(x)$, as Keating and Snaith [19] did in their paper.

Corollary 4.7.4.3. *We have for $s_1, s_2 \in \mathbb{R}, x \in \mathbb{C}$ with* $|x| < 1$

$$\mathbb{E}\left[|Z_n(x)|^{s_1} e^{is_2 \arg(Z_n(x))}\right] \to \prod_{\substack{k_1,k_2\in\mathbb{N}\\ k_1+k_2\neq 0}}^{\infty} \left(1 - x^{k_1}\overline{x}^{k_2}\right)^{\left(\binom{\frac{s_1-s_2}{2}}{k_1}\binom{\frac{s_1+s_2}{2}}{k_2}(-1)^{k_1+1}\right)}.$$

Proof. We have $|z|^{s_1} = z^{s_1/2}\overline{z}^{s_1/2}$ and $e^{is_2 arg(z)} = \frac{z^{s_2}}{|z|^{s_2}}$. □

Chapter 5

Associated class functions

This section is a joint work with Dr. Paul-Olivier Dehaye and based on the paper "*On averages of randomized class functions on the symmetric groups and their asymptotics*" [11].

We study in this section the Moments of $W^n(f)$ for polynomials and holomorphic functions near 0. We also introduce some additional randomization of $W^n(f)$. We begin with the polynomial case and then go to the holomorphic case. We also give the promised probabilistic proof of theorem 4.5.1 and theorem 4.7.4.

5.1 Generating function for $\mathbb{E}\left[W^n(P)\right]$ and P a polynomial

5.1.1 Definition

Remember we have defined in section 3

$$W^n(f)(\sigma) = \prod_{m=1}^{n} f(x^m)^{C_m^{(n)}} = \prod_{i=1}^{l(\lambda)} f(x^{\lambda_i}),$$

with $\lambda = (\lambda_1, \cdots, \lambda_l)$ the cycle-type of σ.
The first formulation with $C_m^{(n)}$ was good to obtain a central limit theorem for $W^n(f)$ (see section 3). We use in this section both formulations. We use that with partitions λ to write down generating functions and that with $C_m^{(n)}$ in the probabilistic proof of theorem 4.5.1 and theorem 4.7.4.

The definition of $W^n(f)$ was for a function $f : S^1 \to \mathbb{C}$. We extend the definition of $W^n(f)$ to the case of several variables $x_1, \cdots, x_p$ with

$$W^n(f)(x_1, \cdots, x_p) = \prod_{m=1}^{l(\lambda)} f(x_1^{\lambda_m}, \cdots, x_p^{\lambda_m}). \tag{5.1.1}$$

Obviously it is not necessary that $|x_j| = 1$. We will use in fact most of the time $|x_j| < 1$. For simplicity, we later restrict the statements and proofs to the case $p = 2$, but all presented results are in fact true for arbitrary p.

5.1.2 Generating function for $\mathbb{E}\left[W^n(P)\right]$

We now write down the generating function for $\mathbb{E}\left[W^n(P)(x_1, x_2)\right]$ for P a polynomial and $|x_j| \le 1$.
We have found in section 2.1.1 that

$$\mathbb{E}_{S_n}\left[u\right] = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} u(\lambda).$$

for each class function $u : S_n \to \mathbb{C}$. The z_λ here is the same z_λ appearing in lemma 2.6.8. We can therefore use lemma 2.6.8 to write down the generating function of $\mathbb{E}\left[W^n(P)(x_1, x_2)\right]$. We get

Lemma 5.1.1. *Let $P(x_1, x_2)$ be a polynomial with*

$$P(x_1, x_2) = \sum_{k_1, k_2=0}^{\infty} b_{k_1,k_2} x_1^{k_1} x_2^{k_2}$$

and $W^n(P)$ its associated class function. We have

$$\sum_{n=0}^{\infty} \mathbb{E}\left[W^n(P)\right] t^n = \sum_{\lambda} \frac{1}{z_\lambda} W^n(P)(\lambda) t^{|\lambda|} = \prod_{k_1=0}^{\infty} \prod_{k_2=0}^{\infty} (1 - x_1^{k_1} x_2^{k_2} t)^{-b_{k_1,k_2}}, \tag{5.1.2}$$

and both sides of (5.1.2) are holomorphic for $|t| < 1$ and $|x_i| \leq 1$. We use the principal branch of logarithm to define z^s for $z \in \mathbb{C} \setminus \mathbb{R}_-$.

Theorem 4.4.1 is now a corollary of this theorem since $(1-x)^s = \sum_{k=0}^{s} \binom{s}{k} (-1)^k x^k$ for $s \in \mathbb{N}$. We have first found the proof with representation theory for theorem 4.4.1.

Proof. We use lemma 2.6.8 with $a_m = P(x_1^m, x_2^m)$ and get

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathbb{E}\left[W^n(P)\right] t^n &= \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \frac{1}{z_\lambda} W^n(P)(\lambda) t^n = \sum_{\lambda} \frac{1}{z_\lambda} \prod_{m=1}^{l(\lambda)} P(x_1^{\lambda_m}, x_2^{\lambda_m}) t^{|\lambda|} \\
&= \exp\left(\sum_{m=1} \frac{1}{m} P(x_1^m, x_2^m) t^m \right) = \exp\left(\sum_{k_1,k_2=0}^{\infty} b_{k_1,k_2} \sum_{m=1} \frac{t^m}{m} (x_1^{k_1} x_2^{k_2})^m \right) \\
&= \exp\left(\sum_{k_1,k_2=0}^{\infty} b_{k_1,k_2} (-1) \log(1 - x_1^{k_1} x_2^{k_2} t) \right) = \prod_{k_1,k_2=0}^{\infty} (1 - x_1^{k_1} x_2^{k_2} t)^{-b_{k_1,k_2}}.
\end{aligned}$$

Unlike later, the exchange of the sums in the second equality is immediate as only finitely many $b_{k_1,k_2} \neq 0$. □

5.2 Definition of randomized class functions for polynomials

In the special case of permutation matrices, we have at least two options: we could replace all the 1s with iid variables or only introduce one new iid variable for each cycle. We describe here the two possibilities, starting with the second option.

5.2.1 One new variable per cycle

Definition 5.2.1. *Let θ be a random variable with values in S^1. We set for $\lambda \vdash n$ and $g \in C_\lambda$*

$$W^1 Z_n(x) = W^1_\theta Z_n(x)(g) := \prod_{m=1}^{l(\lambda)} (1 - \theta_m x^{\lambda_m}), \tag{5.2.1}$$

with $\theta_m \stackrel{d}{=} \theta$, θ_m iid and independent of g.

The fact that this corresponds to introducing one new variable per cycle can be deduced by comparison with (3.1.6). We use the letter W because we originally thought of this as a characteristic polynomial of the wreath product $S^1 \wr S_n$ in the special case $f(x) = 1 - x$. We generalize this to arbitrary polynomials in $\mathbb{C}[x_1, x_2]$.

Definition 5.2.2. *Let θ and ϑ be random variables with values in S^1 and P be a polynomial with*

$$P(x_1, x_2) = \sum_{k_1, k_2 = 0}^{\infty} b_{k_1, k_2} x_1^{k_1} x_2^{k_2}.$$

We set for $g \in C_\lambda$

$$W^{1,n}(P)(x_1, x_2) = W^{1,n}_{\theta,\vartheta}(P)(x_1, x_2)(g) := \prod_{m=1}^{l(\lambda)} P\left(\theta_m x_1^{\lambda_m}, \vartheta_m x_2^{\lambda_m}\right) \tag{5.2.2}$$

with $(\theta_m, \vartheta_m) \stackrel{d}{=} (\theta, \vartheta)$, (θ_m, ϑ_m) iid and independent of g. This defines the first randomized class function *(of the variable g)* associated to *the polynomial P. We also set*

$$W^{1,n}(P_1, P_2)(x_1, x_2) := W^{1,n}(P)(x_1, x_2) \tag{5.2.3}$$

with $P(x_1, x_2) := P_1(x_1) P_2(x_2)$.

We are primarily interested in class functions of the form $W^1(P_1, P_2)(x_1, x_2)$. We have introduced this more complicated definition because we need it in section 5.8.

5.2.2 One new variable per point

Let D be a $n \times n$ diagonal matrix with iid variables θ_i on the diagonal.

Definition 5.2.3. *We set for $g \in \mathcal{S}_n$*

$$W^2 Z_n(x) = W^2_\theta Z_n(x) = W^2_\theta Z_n(x)(g) := \det(I - xDg). \tag{5.2.4}$$

An explicit computation shows that

$$W^2 Z_n(x) = \prod_{m=1}^{l(\lambda)} \left(1 - x^{\lambda_m} \prod_{i=1}^{\lambda_m} \theta_i^{(m)}\right) \text{ for } g \in C_\lambda \tag{5.2.5}$$

with $\left\{\theta_i^{(m)}; 1 \le m \le l(\lambda), 1 \le i \le \lambda_m\right\} = \{\theta_1, \cdots, \theta_n\}$.
This again generalizes to arbitrary polynomials.

Definition 5.2.4. *Let θ and ϑ be a random variables with values in S^1 and P be polynomials with*

$$P(x_1, x_2) = \sum_{k_1, k_2 = 0}^{\infty} b_{k_1, k_2} x_1^{k_1} x_2^{k_2}$$

We set for $g \in C_\lambda$

$$W^{2,n}(P)(x_1, x_2) = W^{2,n}_{\theta,\vartheta}(P)(x_1, x_2)(g) := \prod_{m=1}^{l(\lambda)} P\left(x_1^{\lambda_m} \prod_{i=1}^{\lambda_m} \theta_i^{(m)}, x_2^{\lambda_m} \prod_{i=1}^{\lambda_m} \vartheta_i^{(m)}\right) \tag{5.2.6}$$

with $(\theta_i^{(m)}, \vartheta_i^{(m)}) \stackrel{d}{=} (\theta, \vartheta)$, $(\theta_i^{(m)}, \vartheta_i^{(m)})$ iid and independent of g. This defines the second randomized class function *(of the variable g)* associated to *the polynomial P. We also define*

$$W^{2,n}(P_1, P_2)(x_1, x_2) := W^{2,n}(P)(x_1, x_2), \tag{5.2.7}$$

with $P(x_1, x_2) := P_1(x_1) P_2(x_2)$.

5.3 Generating functions for $W^{1,n}$ and $W^{2,n}$ for polynomials

We prove in this subsection

Theorem 5.3.1. *Let θ and ϑ be random variables with values in S^1 and P be a polynomial with*

$$P(x_1,x_2)=\sum_{k_1,k_2=0}^{\infty} b_{k_1,k_2} x_1^{k_1} x_2^{k_2}.$$

We define

$$\alpha_{k_1,k_2} := \mathbb{E}\left[\theta^{k_1}\vartheta^{k_2}\right]. \tag{5.3.1}$$

Then,

$$\mathbb{E}\left[W^{1,n}(P)(x_1,x_2)\right] = \left[\prod_{k_1,k_2=0}^{\infty}(1-x_1^{k_1}x_2^{k_2}t)^{-b_{k_1,k_2}\alpha_{k_1,k_2}}\right]_n \tag{5.3.2}$$

$$\mathbb{E}\left[W^{2,n}(P)(x_1,x_2)\right] = \left[\prod_{k_1,k_2=0}^{\infty}(1-\alpha_{k_1,k_2}x_1^{k_1}x_2^{k_2}t)^{-b_{k_1,k_2}}\right]_n. \tag{5.3.3}$$

These identities of coefficients can be obtained by expanding formally the (finite) products, but the products in (5.3.2), (5.3.3) are actually holomorphic for $|t|<1$ and $\max\{|x_1|,|x_2|\}\le 1$.

Remark: Thanks to this theorem, we can also compute the generating functions of expressions of the type $\mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}x_1}W^{1,n}(P)(x_1,x_2)\right], \mathbb{E}\left[\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 W^{1,n}(P)(x_1,x_2)\right],\cdots$. One simply has to apply the differential operator to the products in (5.3.2) and (5.3.3), after proving appropriate convergence results.

Proof of theorem 5.5.1. The main ingredients of this proof are equation (2.1.3) and lemma 5.1.1.

Proof of (5.3.2)**.** We first give an expression for $\mathbb{E}\left[W^{1,n}(P)(x_1,x_2)\right]$ with (2.1.3):

$$\mathbb{E}\left[W^{1,n}(P)(x_1,x_2)\right] = \sum_{\lambda\vdash n}\frac{1}{z_\lambda}\mathbb{E}\left[\prod_{m=1}^{l(\lambda)} P\left(\theta_m x_1^{\lambda_m},\vartheta_m x_2^{\lambda_m}\right)\right] \tag{5.3.4}$$

$$= \sum_{\lambda\vdash n}\frac{1}{z_\lambda}\prod_{m=1}^{l(\lambda)}\mathbb{E}\left[P\left(\theta_m x_1^{\lambda_m},\vartheta_m x_2^{\lambda_m}\right)\right]. \tag{5.3.5}$$

We therefore have to calculate $\mathbb{E}\left[P\left(\theta_m x_1^{\lambda_m},\vartheta_m x_2^{\lambda_m}\right)\right]$:

$$\mathbb{E}\left[P\left(\theta_m x_1^{\lambda_m},\vartheta_m x_2^{\lambda_m}\right)\right] = \sum_{k_1,k_2=0}^{\infty} b_{k_1,k_2}(x_1^{\lambda_m})^{k_1}(x_2^{\lambda_m})^{k_2}\mathbb{E}\left[\theta_m^{k_1}\vartheta_m^{k_2}\right]$$

$$= \sum_{k_1,k_2=0}^{\infty} b_{k_1,k_2}\alpha_{k_1,k_2}(x_1^{\lambda_m})^{k_1}(x_2^{\lambda_m})^{k_2}. \tag{5.3.6}$$

We set $f(x_1,x_2) := \sum_{k_1,k_2=0}^{\infty} b_{k_1,k_2}\alpha_{k_1,k_2} x_1^{k_1}x_2^{k_2}$ and get

$$\mathbb{E}\left[W^{1,n}(P)(x_1,x_2)\right] = \sum_{\lambda\vdash n}\frac{1}{z_\lambda}W^n(f)(\lambda). \tag{5.3.7}$$

We now can use lemma 5.1.1 for this f and get

$$\sum_{n=0}^{\infty}\mathbb{E}\left[W^{1,n}(P)(x_1,x_2)\right]t^n = \sum_{n=0}^{\infty}\left(\sum_{\lambda\vdash n}\frac{1}{z_\lambda}W^n(f)(\lambda)\right)t^n = \prod_{k_1,k_2=0}^{\infty}(1-x_1^{k_1}x_2^{k_2}t)^{-b_{k_1,k_2}\alpha_{k_1,k_2}}. \tag{5.3.8}$$

We have therefore found a generating function for $W^{1,n}(P)(x_1,x_2)$.

Proof of (5.3.3)**.** The calculations are very similar. The only difference is that:

$$\begin{aligned}\mathbb{E}\left[P\left(x_1^{\lambda_m}\prod_{i=1}^{\lambda_m}\theta_i^{(m)}, x_2^{\lambda_m}\prod_{i=1}^{\lambda_m}\vartheta_i^{(m)}\right)\right] &= \sum_{k_1,k_2=0}^{\infty} b_{k_1,k_2}(x_1^{\lambda_m})^{k_1}(x_2^{\lambda_m})^{k_2}\prod_{i=1}^{\lambda_m}\mathbb{E}\left[(\theta_i^{(m)})^{k_1}(\vartheta_i^{(m)})^{k_2}\right]\\ &= \sum_{k_1,k_2=0}^{\infty} b_{k_1,k_2}(x_1^{\lambda_m})^{k_1}(x_2^{\lambda_m})^{k_2}\alpha_{k_1,k_2}^{\lambda_m},\end{aligned}$$

with α_{k_1,k_2} as above. Since the exponent of $\alpha_{k_1,k_2}^{\lambda_m}$ is dependent on λ_m, we have to use the several (i.e. more-than-2) variables case of lemma 5.1.1. Explicitly, we use

$$f(x_1,x_2,\alpha_{1,1},\cdots,\alpha_{d_1,d_2}) = \sum_{k_1,k_2=0}^{\infty} b_{k_1,k_2}x_1^{k_1}x_2^{k_2}\alpha_{k_1,k_2},$$

where d_1, d_2 is the degree of P in x_1, x_2. The only monomials in f with non-zero coefficients have the form $x_1^{k_1}x_2^{k_2}\alpha_{k_1,k_2}$. Therefore,

$$\sum_{n=0}^{\infty}\mathbb{E}\left[W^{2,n}(P)(x_1,x_2)\right]t^n = \prod_{k_1,k_2=0}^{\infty}(1-x_1^{k_1}x_2^{k_2}\alpha_{k_1,k_2}t)^{-b_{k_1,k_2}}. \tag{5.3.9}$$

We have thus found a generating function for $W^{2,n}(P)(x_1,x_2)$.

□

5.4 Examples

We give now some examples of generating functions that can be obtained through these results.

5.4.1 The characteristic polynomial and $\theta \equiv \vartheta \equiv 1$

We now can write down the generating function of $\mathbb{E}\left[Z_n^{s_1}(x_1)Z_n^{s_2}(x_2)\right]$ without representation theory. We set $P_1(x_1) = (1-x)^{s_1}$ and $P_2(x_2) = (1-x)^{s_2}$. Clearly $\alpha_{k_1,k_2} = 1$, so $W^1 = W^2$ (this of course needs not be true in general). We get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{E}\left[Z_n^{s_1}(x_1)Z_n^{s_2}(x_2)\right]t^n &= \sum_{n=0}^{\infty}\mathbb{E}\left[W^{1,n}(P_1,P_2)(x_1,x_2)\right]t^n = \sum_{n=0}^{\infty}\mathbb{E}\left[W^{2,n}(P_1,P_2)(x_1,x_2)\right]t^n\\ &= \prod_{k_1=0}^{s_1}\prod_{k_2=0}^{s_2}\left(1-x_1^{k_1}x_2^{k_2}t\right)^{\binom{s_1}{k_1}\binom{s_2}{k_2}(-1)^{k_1+k_2+1}}\end{aligned} \tag{5.4.1}$$

This reproves theorem 4.4.2.

5.4.2 The case $\theta = \overline{\vartheta}$ uniform on S^1

We have $\alpha_{k_1,k_2} = \begin{cases} 1, & k_1 = k_2 \\ 0, & \text{otherwise.} \end{cases}$, which again implies $W^1 = W^2$. We get for

$$P_1(x_1) = \sum_{k=0}^{d_1} a_k x^k, \qquad P_2(x_2) = \sum_{k=0}^{d_2} b_k x^k$$

that

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbb{E}\left[W^{1,n}(P_1,P_2)(x_1,x_2)\right] t^n &= \sum_{n=0}^{\infty} \mathbb{E}\left[W^{2,n}(P_1,P_2)(x_1,x_2)\right] t^n \\ &= \prod_{k=0}^{\infty} (1 - x_1 x_2 t)^{-a_k b_k} \end{aligned} \tag{5.4.2}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbb{E}\left[\left(W^1 Z_n(x)\right)^{s_1} \overline{\left(W^1 Z_n(x)\right)^{s_2}}\right] t^n &= \sum_{n=0}^{\infty} \mathbb{E}\left[\left(W^2 Z_n(x)\right)^{s_1} \overline{\left(W^2 Z_n(x)\right)^{s_2}}\right] t^n \\ &= \prod_{k=0}^{\infty} (1 - |x|^{2k} t)^{-\binom{s_1}{k}\binom{s_2}{k}}. \end{aligned} \tag{5.4.3}$$

Equation (5.4.3) is also valid for $|x| = 1$ (see theorem 5.3.1). We get in this case

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbb{E}\left[\left(W^1 Z_n(x)\right)^{s_1} \overline{\left(W^1 Z_n(x)\right)^{s_2}}\right] t^n &= \sum_{n=0}^{\infty} \mathbb{E}\left[\left(W^2 Z_n(x)\right)^{s_1} \overline{\left(W^2 Z_n(x)\right)^{s_2}}\right] t^n \\ &= (1-t)^{-\binom{s_1+s_2}{s_1}}, \end{aligned} \tag{5.4.4}$$

where we have used the Vandermonde identity for binomial coefficients.

5.4.3 An example with $\theta = \overline{\vartheta}$ discrete on S^1

We choose θ with $\mathbb{P}\left[\theta = e^{m\frac{2\pi i}{p}}\right] = \frac{1}{p}$ for $p \in \mathbb{N}$ and $\vartheta = \overline{\theta}$. Then $\alpha_{k_1,k_2} = \begin{cases} 1, & p \text{ divides } (k_1 - k_2) \\ 0, & \text{otherwise} \end{cases}$, and still $W^1 = W^2$:

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbb{E}\left[W^{1,n}(P_1,P_2)(x_1,x_2)\right] t^n &= \sum_{n=0}^{\infty} \mathbb{E}\left[W^{2,n}(P_1,P_2)(x_1,x_2)\right] t^n \\ &= \prod_{\substack{k_1,k_2=0 \\ p|(k_1-k_2)}}^{\infty} (1 - x_1 x_2 t)^{-a_{k_1} b_{k_2}}. \end{aligned} \tag{5.4.5}$$

This situation is similar to what is described in [26].

5.5 Asymptotics of $\mathbb{E}\left[W^{j,n}(P)\right]$ for $|x| < 1$ and P a polynomial

We have found in section 5.3 generating functions for both types of class functions. We can now extract the behavior of $\mathbb{E}\left[W^{j,n}(P)\right]$ for $n \to \infty$ and $\max\{|x_1|, |x_2|\} < 1$.

Theorem 5.5.1. *Let x_1, x_2 be complex numbers with $|x_i| < 1$ and P be a polynomial with*

$$P(x_1,x_2) = \sum_{k_1,k_2=0}^{\infty} b_{k_1,k_2} x_1^{k_1} x_2^{k_2}.$$

If $b_{0,0} \notin \mathbb{Z}_{\leq 0}$ then

$$\mathbb{E}\left[W^{1,n}(P)(x_1,x_2)\right] \sim \frac{n^{b_{0,0}-1}}{\Gamma(b_{0,0})} \prod_{\substack{k_1,k_2\in\mathbb{N}\\ k_1+k_2\neq 0}} (1 - x_1^{k_1} x_2^{k_2})^{-b_{k_1,k_2}\alpha_{k_1,k_2}} \tag{5.5.1}$$

and

$$\mathbb{E}\left[W^{2,n}(P)(x_1,x_2)\right] \sim \frac{n^{b_{0,0}-1}}{\Gamma(b_{0,0})} \prod_{\substack{k_1,k_2\in\mathbb{N}\\ k_1+k_2\neq 0}}^{\infty} (1 - \alpha_{k_1,k_2} x_1^{k_1} x_2^{k_2})^{-b_{k_1,k_2}}. \tag{5.5.2}$$

If $b_{0,0} \in \mathbb{Z}_{\leq 0}$ then we just have

$$\mathbb{E}\left[W^{1,n}(P)(x_1,x_2)\right] \to 0, \tag{5.5.3}$$
$$\mathbb{E}\left[W^{2,n}(P)(x_1,x_2)\right] \to 0. \tag{5.5.4}$$

Proof. One can prove this theorem by induction over the number of factors. Since the exponents are now complex, the calculations are much more technical than the calculations in the proof of theorem 4.5.1. We therefore omit this proof. One can of course also use theorem 2.6.6 since there are only finitely many singularities on S^1. We give in section 5.8 a probabilistic proof.

□

5.6 Asymptotics of $\mathbb{E}\left[W^{j,n}(P)\right]$ for $|x| = 1$ and P a polynomial

The exponents in the generating functions in theorem 5.3.1 can now be complex. Therefore it is not anymore possible to use a partial fraction decomposition as in section 4.6. But the generating functions in theorem 5.3.1 are finite products. Thus we can use theorem 2.6.6 to calculate the asymptotic behavior of $\mathbb{E}\left[W^{j,n}(P)\right]$ for $|x_1| = |x_2| = 1$.
One simply has to check the singularities of the generating function. Since there exists several different situations we give only one particular example.

Theorem 5.6.1. *Let $P(x_1,x_2) = \sum_{k_1,k_2=0}^{\infty} b_{k_1,k_2} x_1^{k_1} x_2^{k_2}$ be a polynomial, $x \in S^1$ and assume that $|\alpha_{k_1,k_2}| < 1$ for $k_1 + k_2 > 0$.*
If $b_{0,0} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ then

$$\mathbb{E}\left[W^{2,n}(P)(x)\right] \sim \frac{n^{b_{0,0}-1}}{\Gamma(b_{0,0})} \left(\prod_{k_1=0}^{\infty} \prod_{k_2=0}^{\infty}\right)' (1 - \alpha_{k_1,k_2} x_1^{k_1} x_2^{k_2})^{-b_{k_1,k_2}}. \tag{5.6.1}$$

Proof. Since $|\alpha_{k_1,k_2}| < 1$ for $k_1 + k_2 > 0$, the generating function of $W^{2,n}$ is holomorphic inside the unit disc and has only 1 as singularity on S^1. The theorem then follows form theorem 2.6.6. □

5.7 Randomized Class Functions associated to Holomorphic Functions

Our goal is now to extend what we did for polynomials onto holomorphic functions. The definition of $W^n(P)$ in (3.1.7) or (3.1.9) directly generalizes to holomorphic functions. One only has to be more careful with the domain of definition. The proofs for holomorphic functions thus apply to section 5.3 and section 5.5 as well, but they are more challenging technically: the products that were finite now become infinite, which require us to leap beyond formal generating series in t and actually consider convergence issues in t.
The main issue arises with the extension of theorem 5.5.1: The theorem is also true for holomorphic functions, but we cannot argue anymore by induction over the number of factors, since there are now infinitely many. We can still use theorem 2.6.6. We will give a different proof with probability theory. Since this needs a lot of work, we defer the extension of theorem 5.5.1 to section 5.8.
Let $x_0 \in \mathbb{C}$ and $r \in \mathbb{R}_+$, and set $B_r(x_0) := \{x \in \mathbb{C}; |x - x_0| < r\}$. We now extend lemma 5.1.1 (again stated for the case of $p = 2$ variables only):

Lemma 5.7.1. *Let $f(x_1, x_2)$ be a holomorphic function in $B_{r_1}(0) \times B_{r_2}(0)$ with*

$$f(x_1, x_2) = \sum_{k_1,k_2=0}^{\infty} b_{k_1,k_2} x_1^{k_1} x_2^{k_2}$$

and $W^n(f)$ its associated class function. Set

$$\Omega' := \left\{(x_1, x_2) \in \mathbb{C}^2; |x_i| \leq 1 \text{ if } r_i > 1 \text{ and } |x_i| < r_i \text{ if } r_i \leq 1\right\}.$$

We then have on $\Omega' \times B_1(0)$

$$\sum_{\lambda} \frac{1}{z_\lambda} W^n(f)(\lambda) t^n = \prod_{k_1=0}^{\infty} \prod_{k_2=0}^{\infty} (1 - x_1^{k_1} x_2^{k_2} t)^{-b_{k_1,k_2}}. \tag{5.7.1}$$

The product is holomorphic in (x_1, x_2, t) in the interior of $\Omega' \times B_1(0)$. The product is holomorphic in t in $B_1(0)$ for all $(x_1, x_2) \in \Omega'$. Note that we use the principal branch of logarithm to define z^s for $z \in \mathbb{C} \setminus \mathbb{R}_-$.

Proof. The proof works as the proof of lemma 5.1.1, except that the justification for the exchange of sums that occurs in the second equality is barely more tricky:

$$\begin{aligned}\left|\sum_{k_1,k_2=0}^{\infty} \sum_{m=1}^{\infty} \frac{t^m}{m} b_{k_1,k_2} (x_1^{k_1} x_2^{k_2})^m\right| &\leq \sum_{k_1,k_2=0}^{\infty} \sum_{m=1}^{\infty} \left|\frac{t^m}{m}\right| |b_{k_1,k_2} x_1^{k_1} x_2^{k_2}| \\ &\leq \log(1 - |t|) \sum_{k_1,k_2=0}^{\infty} |b_{k_1,k_2} x_1^{k_1} x_2^{k_2}| < \infty,\end{aligned}$$

where the last inequality is true since f is holomorphic in $B_{r_1}(0) \times B_{r_2}(0)$. □

Remark: The conditions on x_1, x_2 and t ensures that the Taylor-expansion of $\log(1 - x_1^{k_1} x_2^{k_2} t)$ is absolutely convergent. If f is a meromorphic function, one can replace this condition by any other that ensures $\sup\left\{|x_1^{k_1} x_2^{k_2} t|, b_{k_1,k_2} \neq 0\right\} < 1$.

Until now, we have defined randomized class functions of two types associated to polynomials. We can use formula (5.2.2) and (5.2.6) to define $W^{j,n}(f)(x_1, x_2)$ for a holomorphic

function f. We will always assume in what follows that f is holomorphic in $B_{r_1}(0)\times B_{r_2}(0)$. The function $W^{j,n}(f)(x_1,x_2)$ is also holomorphic in $B_{r_1}(0)\times B_{r_2}(0)$ since $\theta,\vartheta\in S^1$.
We now get a complete analog of the result in section 5.3.

Theorem 5.7.2. *Let f be a holomorphic function in $B_{r_1}(0)\times B_{r_2}(0)$ with*

$$f(x_1,x_2)=\sum_{k_1,k_2=0}^{\infty} b_{k_1,k_2}x_1^{k_1}x_2^{k_2}$$

We define as in (5.3.1) $\alpha_{k_1,k_2}:=\mathbb{E}\left[\theta^{k_1}\vartheta^{k_2}\right]$ and

$$\Omega':=\left\{(x_1,x_2)\in\mathbb{C}^2;|x_i|\le 1 \text{ if } r_i>1 \text{ and } |x_i|<r_i \text{ if } r_i\le 1\right\}.$$

We get

$$\mathbb{E}\left[W^{1,n}(f)(x_1,x_2)\right]=\left[\prod_{k_1,k_2=0}^{\infty}(1-x_1^{k_1}x_2^{k_2}t)^{-\alpha_{k_1,k_2}b_{k_1,k_2}}\right]_n \tag{5.7.2}$$

$$\mathbb{E}\left[W^{2,n}(f)(x_1,x_2)\right]=\left[\prod_{k_1,k_2=0}^{\infty}(1-\alpha_{k_1,k_2}x_1^{k_1}x_2^{k_2}t)^{-b_{k_1,k_2}}\right]_n. \tag{5.7.3}$$

The products are holomorphic in (x_1,x_2,t) in the interior of $\Omega'\times B_1(0)$. The products are holomorphic in t in $B_1(0)$ for all $(x_1,x_2)\in\Omega'$.

Proof. The proof of (5.7.2) is almost similar to the proof of (5.3.2). There is only one important difference. The function f defined in the proof of (5.3.2) is now a holomorphic function and not a polynomial. We thus have to apply lemma 5.7.1 instead of lemma 5.1.1. The function f is holomorphic in $B_{r_1}(0)\times B_{r_2}(0)$ since $|\alpha_{k_1,k_2}|\le 1$. This is thus sufficient to prove (5.7.2).
The proof of (5.7.3) is little bit more intricate. We cannot keep the analogy with the proof of (5.3.3) and simply use lemma 5.7.1. Indeed, we would now have a holomorphic function in infinitely many variables. Thankfully, we can still use our main lemma, lemma 2.6.8. We have found in the proof of lemma 5.7.1 that

$$\mathbb{E}\left[P\left(x_1^{\lambda_m}\prod_{i=1}^{\lambda_m}\theta_i^{(m)},x_2^{\lambda_m}\prod_{i=1}^{\lambda_m}\vartheta_i^{(m)}\right)\right]=\sum_{k_1,k_2=0}^{\infty}b_{k_1,k_2}(x_1^{\lambda_m})^{k_1}(x_2^{\lambda_m})^{k_2}\alpha_{k_1,k_2}^{\lambda_m}.$$

We define here

$$a_m:=\sum_{k_1,k_2=0}^{\infty}b_{k_1,k_2}(x_1^m)^{k_1}(x_2^m)^{k_2}\alpha_{k_1,k_2}^m$$

and get with lemma 2.6.8

$$\sum_{n=0}^{\infty}\left(\mathbb{E}\left[W^{2,n}(f)(x_1,x_2)\right]\right)t^n=\sum_{n=0}^{\infty}\left(\sum_{\lambda\vdash n}\frac{1}{z_\lambda}a_\lambda\right)t^n=\exp\left(\sum_{m=1}^{\infty}a_mt^m\right).$$

The last steps are now completely similar to the proof of lemma 5.7.1. $\square$

We now consider the generalization of theorem 5.5.1 to the case of holomorphic functions.

5.8 Asymptotics for randomized class functions associated to holomorphic functions

We assume as in the last section that f is holomorphic in $B_{r_1}(0) \times B_{r_2}(0)$. We have calculated in theorem 5.5.1 the behavior of $\mathbb{E}\left[W^{j,n}(P)\right]$ for $n \to \infty$. It is natural to ask if this lemma generalizes to class functions associated to holomorphic functions. Explicitly, we prove

Theorem 5.8.1. *Let $x_1, x_2 \in \mathbb{C}$ be given with $|x_i| < \min(r_i, 1)$ and f, α_{k_1,k_2} be as in theorem 5.7.2.*
If $b_{0,0} \notin \mathbb{Z}_{\leq 0}$, then

$$\mathbb{E}\left[W^{1,n}(f)(x_1,x_2)\right] \sim \frac{n^{b_{0,0}-1}}{\Gamma(b_{0,0})} \prod_{\substack{k_1,k_2 \in \mathbb{N} \\ k_1+k_2 \neq 0}} (1 - x_1^{k_1} x_2^{k_2})^{-b_{k_1,k_2}\alpha_{k_1,k_2}} \tag{5.8.1}$$

and

$$\mathbb{E}\left[W^{2,n}(f)(x_1,x_2)\right] \sim \frac{n^{b_{0,0}-1}}{\Gamma(b_{0,0})} \prod_{\substack{k_1,k_2 \in \mathbb{N} \\ k_1+k_2 \neq 0}} (1 - \alpha_{k_1,k_2} x_1^{k_1} x_2^{k_2})^{-b_{k_1,k_2}}. \tag{5.8.2}$$

If $b_{0,0} \in \mathbb{Z}_{\leq 0}$ then there exists a $\delta = \delta(x_1,x_2) > 0$ such that

$$\mathbb{E}\left[W^{1,n}(f)(x_1,x_2)\right] = O\left(\frac{1}{(1+\delta)^n}\right) \qquad (n \to \infty), \tag{5.8.3}$$

$$\mathbb{E}\left[W^{2,n}(f)(x_1,x_2)\right] = O\left(\frac{1}{(1+\delta)^n}\right) \qquad (n \to \infty). \tag{5.8.4}$$

This theorem is a direct generalization of theorem 5.5.1. We cannot argue anymore by induction over the number of factors, since there are infinitely many factors in the RHS of (5.5.1) and (5.5.2). One still can use theorem 2.6.6, but we give here a different proof with probability theory. The main argumentation will base on the Feller coupling. This proof will occupy us for this whole section.
The proof of theorem 5.8.1 will run as follows. We prove in section 5.8.1 that the case $b_{0,0} = 1$ implies the general case. We first prove theorem 5.8.1 for $W^{1,n}(f)$ for $b_{0,0} = 1$ and give at the end some comments for the proof for $W^{2,n}$. In order to prove this, we do in section 5.8.2 some simplifications and made some small conventions. After these preparations, we suggest in section 5.8.3 a candidate $W^{1,\infty}(f)$ for the limit in n of $W^{1,n}(f)$ and prove some analytic results on it. Finally, we prove in section 5.8.4 the convergence $\mathbb{E}\left[W^{1,n}(f)\right] \to \mathbb{E}\left[W^{1,\infty}(f)\right]$.

5.8.1 Reduction to $b_{0,0} = 1$

Lemma 5.8.2. *If theorem 5.8.1 is true for $b_{0,0} = 1$ then it is true for all $b_{0,0} \in \mathbb{C}$.*

Before we prove this lemma, we do some (small) preparations.

Remember: we have defined in (4.7.1)

$$\binom{s}{k} := \prod_{m=1}^{k} \frac{s-m+1}{m} = \frac{\Gamma(s+1)}{\Gamma(k+1)\Gamma(s-k+1)} \text{ and } \binom{s}{0} := 1. \tag{5.8.5}$$

for $s \in \mathbb{C}, k \in \mathbb{N}$. The proof of the last equality can be found in [14]. We then have

Lemma 5.8.3. *We have for each $s, z \in \mathbb{C}$ with $|z| < 1$ or $\mathrm{Re}(s) < 0, |z| = 1$ that*

$$\frac{1}{(1+z)^s} = \sum_{k=0}^{\infty} \binom{s-1+k}{k} z^k, \tag{5.8.6}$$

and the sum is absolutely convergent in both cases. Also, we have for $s \notin \{-1, -2, -3, \cdots\}$

$$\binom{n+s}{n} = \frac{n^s}{\Gamma(s+1)}\big(1 + O(n^{-1})\big) \qquad (n \to \infty). \tag{5.8.7}$$

We also need

Lemma 5.8.4 (Euler-MacLaurin-Formel, see [3])**.** *Let $a : [0, \infty] \to \mathbb{C}$ be smooth function. We then have for all $n \geq 2$*

$$\sum_{k=2}^{n} a(k) = \int_1^n a(s)ds + \frac{a(n) - a(1)}{2} - \int_1^n (s - \lfloor s \rfloor) a'(s) ds \tag{5.8.8}$$

Proof of lemma 5.8.2. We prove this lemma only for W^1, since the proof for W^2 is the same. We put $c = b_{0,0} - 1$ and rewrite the generating function in (5.7.2) as follows:

$$\left(\frac{1}{(1-t)^c}\right) \underbrace{\left(\frac{1}{1-t} \prod_{\substack{k_1,k_2=0 \\ k_1+k_2 \neq 0}}^{\infty} (1 - x_1^{k_1} x_2^{k_2} t)^{-b_{k_1,k_2} \alpha_{k_1,k_2}}\right)}_{=: h(x_1,x_2,t)}. \tag{5.8.9}$$

We define $\widetilde{f}(x_1, x_2) := f(x_1, x_2) - c$. Then $\widetilde{f}(0, 0) = 1$ and $h(x_1, x_2, t)$ is the generating function for $W^1(\widetilde{f})(x_1, x_2)$ (compare with (5.7.2)). Since we assume that theorem 5.8.1 is true for $b_{0,0} = 1$ and in that case the RHS of (5.8.1) does not depend on n, we can write $h(x_1, x_2, t) = \sum_{n=0}^{\infty} h_n t^n$ with $h_n \to h_\infty \in \mathbb{C}$. We first look at the convergence rate of the sequence $(h_k)_{k \in \mathbb{N}}$. Let x_1, x_2 be fixed. The function $(1-t)h(x_1, x_2, t)$ is holomorphic in $B_{1+\delta}(0)$ for some $\delta > 0$ small enough. This can be seen by inspecting the proof of lemma 5.7.1. The convergence radius of the expansion of $(1-t)h(x_1, x_2, t)$ around 0 is therefore at least $1 + \delta$ and so $|\left[(1-t)h(x_1, x_2, t)\right]_n| = O\big((1+\delta)^{-n}\big)$. But $\left[(1-t)h(x_1, x_2, t)\right]_n = h_n - h_{n-1}$. Therefore $|h_n - h_{n-1}| = O\big((1+\delta)^{-n}\big)$. We get

$$\begin{aligned} |h_\infty - h_n| &= \lim_{m\to\infty} |h_m - h_n| \leq \lim_{m\to\infty} \sum_{k=n+1}^{m} |h_k - h_{k-1}| = O\left(\sum_{k=n+1}^{m} (1+\delta)^{-k}\right) \\ &\leq O\left(\sum_{k=n+1}^{\infty} (1+\delta)^{-k}\right) = O\big((1+\delta)^{-n}\big) \end{aligned} \tag{5.8.10}$$

We are ready to proof theorem 5.8.2.

Case 0, $c = 0$ This case is trivial, since $c = 0$ implies $b_{0,0} = 1$.

Case 1, $c \in \{-1, -2, -3, \cdots\}$ In this case $-c \in \mathbb{N}$ and $\binom{-c}{k} = 0$ for $k \geq -c$. We get for

$n \geq -c$

$$\left[\frac{1}{(1-t)^c} h(x_1, x_2, t)\right]_n = \left[(1-t)^{-c} h(x_1, x_2, t)\right]_n = \sum_{k=0}^{-c} h_{n-k} \binom{-c}{k} (-1)^k$$
$$= \sum_{k=0}^{-c} \Big(h_\infty + O\big((1+\delta)^{-(n-k)}\big)\Big) \binom{-c}{k} (-1)^k$$
$$= h_\infty \left(\sum_{k=0}^{-c} \binom{-c}{k} (-1)^k\right) + O\big((1+\delta)^{-n}\big) = O\big((1+\delta)^{-n}\big)$$

Case 2, $c \notin \mathbb{Z}_{\leq 0}$ This case is a little bit more difficult. We have

$$\left[\frac{1}{(1-t)^c} h(x_1, x_2, t)\right]_n = \left[(1-t)^{-c} h(x_1, x_2, t)\right]_n = \sum_{k=0}^{n} h_{n-k} \binom{c-1+k}{k}$$
$$= h_n + c h_{n-1} + \sum_{k=2}^{n} \Big(h_\infty + O\big((1+\delta)^{-(n-k)}\big)\Big) \Big(\frac{k^{c-1}}{\Gamma(c)} + O\big(k^{c-2}\big)\Big)$$

Obviously we have $h_n + c h_{n-1} = (1+c) h_\infty + O\big(n^{c-2}\big)$.
It follows immediately form lemma 5.8.4 (with a small calculation) that

$$\sum_{k=2}^{n} \frac{k^{c-1}}{\Gamma(c)} = \mathit{const.} + \frac{1}{\Gamma(c+1)} n^c + O\big(n^{c-2}\big) \tag{5.8.11}$$

Therefore the leading therm gives preciously what we want. It rest to show that the other terms behaves well. We get again with lemma 5.8.4 and $d := \operatorname{Re}(c)$

$$\sum_{k=2}^{n} \left|(1+\delta)^{-(n-k)} k^{c-1}\right| = \frac{1}{(1+\delta)^n} \sum_{k=2}^{n} k^{d-1} (1+\delta)^k$$
$$= \frac{1}{(1+\delta)^n} \left(\int_1^n s^{d-1} (1+\delta)^s \, ds + \frac{n^{d-1}(1+\delta)^n - (1-\delta)}{2} \right.$$
$$\left. - \int_1^n (s - \lfloor s \rfloor) \left(s^{d-1} (1+\delta)^s\right)' ds\right).$$

But

$$\left|\int_1^n s^{d-1} (1+\delta)^s \, ds\right| = \left|\left(s^{d-1} \frac{(1+\delta)^s}{\log(1+\delta)}\right)\Big|_{s=1}^n - \int_1^n s^{d-2} \frac{(1+\delta)^s}{\log(1+\delta)} \, ds\right|$$
$$\leq \left(s^{d-1} \frac{(1+\delta)^s}{\log(1+\delta)}\right)\Big|_{s=1}^n + \int_1^n s^{d-2} \frac{(1+\delta)^s}{\log(1+\delta)} \, ds$$
$$\leq \left(n^{d-1} \frac{(1+\delta)^n}{\log(1+\delta)} - \frac{(1+\delta)}{\log(1+\delta)}\right) + \int_1^n s^{d-2} \frac{(1+\delta)^n}{\log(1+\delta)} \, ds$$
$$= \mathit{const.} + O(n^{d-1}(1+\delta)^n) = \mathit{const.} + O(n^{c-1}(1+\delta)^n)$$

Therefore

$$\sum_{k=2}^{n} \left|(1+\delta)^{-(n-k)} k^{c-1}\right| = \mathit{const.} + O(n^{c-1})$$

We put everything together and get

$$\left[\frac{1}{(1-t)^c} h(x_1, x_2, t)\right]_n = \mathit{const.} + \frac{h_\infty}{\Gamma(c+1)} n^c + O(n^{c-1}) \tag{5.8.12}$$

If $\mathrm{Re}(c) > 0$ than (5.8.12) is enough to proof lemma 5.8.2. If $\mathrm{Re}(c) < 0$ then we have to prove that the constant is 0. We know that the sequence $(h_k)_{k\in\mathbb{N}}$ is bounded by a constant C. Therefore

$$\left|\left[\frac{1}{(1-t)^c}h(x_1,x_2,t)\right]_n\right| = \left|\sum_{k=0}^{n} h_{n-k}\binom{c-1+k}{k}\right| \\ \leq C\sum_{k=0}^{n}\left|\binom{c-1+k}{k}\right| \leq C\sum_{k=0}^{\infty}\left|\binom{c-1+k}{k}\right| < \infty$$

by lemma 5.8.3. We therefore can apply dominated convergence (with $h_k = 0$ for $k < 0$) and get

$$\lim_{n\to\infty}\sum_{k=0}^{\infty} h_{n-k}\binom{c-1+k}{k} = \sum_{k=0}^{\infty}\lim_{n\to\infty} h_{n-k}\binom{c-1+k}{k} = \sum_{k=0}^{\infty} h_{\infty}\binom{c-1+k}{k} = 0$$

It rest to show the case $\mathrm{Re}(c) = 0, c \neq 0$. It is easy to that the constant in (5.8.12) is continuous in c. This completes the proof.

□

5.8.2 Simplifications and conventions

We calculate as in (5.3.6):

$$\mathbb{E}\left[f\left(\theta_m x_1, \vartheta_m x_2\right)\right] = \sum_{k_1,k_2=0} b_{k_1,k_2}\alpha_{k_1,k_2}x_1^{k_1}x_2^{k_2} =: \widetilde{f}(x_1,x_2)$$

and therefore

$$\mathbb{E}\left[W^{1,n}_{\theta,\vartheta}(f)(x_1,x_2)\right] = \mathbb{E}\left[W^{1,n}_{1,1}(\widetilde{f})(x_1,x_2)\right] = \mathbb{E}\left[W^{1,n}(\widetilde{f})(x_1,x_2)\right].$$

Since we are only interested in expectations, we can assume $\theta \equiv \vartheta \equiv 1$ and consider $\widetilde{f}$ instead of f.
We will also assume that f only depends on one variable. The arguments in the proof are the same, but the expressions are much more simple.
We now set a $0 < r < \min\{1, r_1\}$ fixed and prove theorem 5.8.1 for $|x| < r$. We shrink x because we sometimes need $\sup|f(x)|$ to be finite.

5.8.3 The limit distribution

In this subsection we write down a possible limit of $W^{1,n}(f)(x)$ and show that it is a good candidate.
If a $\sigma \in \mathcal{C}_\lambda$ with $|\lambda| = n$ is given then

$$W^{1,n}(f)(x) = \prod_{m=1}^{n} f(x^m)^{C_m^{(n)}}.$$

We know form lemma 2.1.10 that $C_m^{(n)} \xrightarrow{d} Y_m$ and so a natural and possible limit would be

$$W^{\infty}(f)(x) := W^{\infty}(f) := \prod_{m=1}^{\infty} f(x^m)^{Y_m}. \tag{5.8.13}$$

We prove in lemma 5.8.8 that $W^n(f) \xrightarrow{d} W^{\infty}(f)$. Of course there are many things we need to check. We start with

Lemma 5.8.5. *The function $W^\infty(f)(x)$ is a.s. a holomorphic function of $x \in B_r(0)$.*

Proof. We first mention that $\log((1-x)^m) \equiv m \log(1-x) \mod 2\pi i$ for $m \in \mathbb{N}$. The argument of log is always in $[-\pi, \pi]$ and therefore

$$|\log((1-x)^m)| \leq m\, |\log(1-x)| \text{ for } m \in \mathbb{N}.$$

We write next $f(x) = 1 + xh(x)$ with h holomorphic in $B_r(0)$. Choose $m_0 \in \mathbb{N}$ such that $|x^m h(x^m)| < r < 1$ for all $m \geq m_0$ and all $x \in B_r(0)$. We define $\delta = \delta(r) = \sup_{x \in B_r(0)} |h(x)|$ and remember the general fact that there exists $\beta = \beta(r)$ with $|\log(1+x)| \leq \beta |x|$ for $|x| < r$. We get

$$\begin{aligned}
\left| \log \left(\prod_{m=m_0}^{\infty} f(x^m)^{Y_m} \right) \right| &\leq \sum_{m=m_0}^{\infty} Y_m |\log\bigl(1 + x^m h(x^m)\bigr)| \\
&\leq \beta \sum_{m=m_0}^{\infty} Y_m |x^m h(x^m)| \leq \delta\beta \sum_{m=m_0}^{\infty} Y_m r^m.
\end{aligned}$$

We prove in lemma 5.8.6 that the last sum is a.s. finite. Therefore $\prod_{m=m_0}^{\infty} f(x^m)^{Y_m}$ is a.s. a holomorphic function in x and so is $W^\infty(f)(x)$.

□

Lemma 5.8.6. *We have $\sum_{m=1}^{\infty} Y_m r^m$ is a.s. absolute convergent for $|r| < 1$.*

Proof. We prove the absolute convergence of the sum $\sum_{m=1}^{\infty} Y_m r^m$ by showing

$$\limsup_{m\to\infty} \sqrt[m]{|Y_m r^m|} < 1 \text{ a.s.}$$

We fix an a with $r < a < 1$ and set $A_m := \{\, Y_m r^m > a^m \} = \left\{ \sqrt[m]{|Y_m r^m|} > a \right\}$. Then

$$\begin{aligned}
\mathbb{P}\left[\limsup_{m\to\infty} \sqrt[m]{|Y_m r^m|} < 1 \right] &\geq 1 - \mathbb{P}\left[\limsup_{m\to\infty} \sqrt[m]{|Y_m r^m|} > a \right] \\
&= 1 - \mathbb{P}\left[\cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_m \right] = 1 - \mathbb{P}\left[\limsup(A_m) \right].
\end{aligned}$$

We get with Markov's inequality

$$\mathbb{P}\left[Y_m > \left(\frac{a}{r}\right)^m \right] \leq \frac{\mathbb{E}[Y_m]}{\left(\frac{a}{r}\right)^m} = \frac{1}{m} \left(\frac{r}{a}\right)^m.$$

Therefore $\sum_m \mathbb{P}\left[Y_m > \left(\frac{a}{r}\right)^m \right] < \infty$.
It follows from the Borel-Cantelli-lemma that $\mathbb{P}[\limsup(A_m)] = 0$. □

We have proven that $W^\infty(f)(x)$ is a.s. a holomorphic function. This does not imply the holomorphicity of $\mathbb{E}[W^\infty(f)(x)]$, even when it exists. We therefore prove

Lemma 5.8.7. *Let $f(x) := \sum_k b_k x^k$ and $x \in B_r(0)$. Then all moments of $W^\infty(f)(x)$ exist. The expectation $\mathbb{E}[W^\infty(f)(x)]$ is a holomorphic function on $B_r(0)$ with*

$$\mathbb{E}\left[W^{1,\infty}(f)(x) \right] = \prod_{k=1}^{\infty} \frac{1}{(1 - x^k)^{b_k}}.$$

Proof. **Step 1** We show that $\mathbb{E}\left[W^\infty(f)(x)\right]$ exists. We define h and δ as above and obtain

$$\mathbb{E}\left[\left|\prod_{m=1}^{\infty} f(x^m)^{Y_m}\right|\right] = \mathbb{E}\left[\prod_{m=1}^{\infty}\left|\big(1+x^m h(x^m)\big)^{Y_m}\right|\right] \leq \mathbb{E}\left[\prod_{m=1}^{\infty}(1+\delta r^m)^{Y_m}\right]$$
$$\stackrel{(*)}{=} \prod_{m=1}^{\infty} \exp\left(\frac{(1+\delta r^m)-1}{m}\right) = \exp\left(\delta\sum_{m=1}^{\infty}\frac{r^m}{m}\right) < \infty,$$

where in $\stackrel{(*)}{=}$ we have used that

$$\mathbb{E}\left[y^{(Y_m)}\right] = \exp\left(\frac{y-1}{m}\right) \text{ for } y \geq 0$$

when Y_m is a Poisson distributed random variable with $\mathbb{E}\left[Y_m\right] = \frac{1}{m}$. This can be shown by a simple calculation, expanding the exponential series. This proves the existence of $\mathbb{E}\left[W^\infty(f)(x)\right]$.

Step 2 We calculate the value of $\mathbb{E}\left[W^\infty(f)(x)\right]$:

$$\mathbb{E}\left[W^\infty(f)(x)\right] = \mathbb{E}\left[\prod_{m=1}^{\infty} f(x^m)^{Y_m}\right] = \prod_{m=1}^{\infty}\mathbb{E}\left[f(x^m)^{Y_m}\right] = \prod_{m=1}^{\infty}\exp\left(\frac{f(x^m)-1}{m}\right)$$
$$= \exp\left(\sum_{m=1}^{\infty}\frac{1}{m}\sum_{k=1}^{\infty} b_k x^{mk}\right) = \exp\left(\sum_{k=1}^{\infty} b_k \sum_{m=1}^{\infty}\frac{x^{km}}{m}\right)$$
$$= \exp\left(\sum_{k=1}^{\infty} b_k\big(-\log(1-x^k)\big)\right) = \prod_{k=1}^{\infty}\frac{1}{(1-x^k)^{b_k}}.$$

The exchange of the exponential and the product in the first line is justified by step 1. The exchange of the two sums is justified as follows: the convergence radius of the Taylor expansion of f is at least r_1 since f is holomorphic in $B_{r_1}(0)$. Since we have chosen $r < r_1$ we get

$$\limsup_{k\to\infty} |b_k|^{1/k} \leq \frac{1}{r_1} < \frac{1}{r}$$

Therefore there exists a constant C with $|b_k| < C(\frac{1}{r_1})^k$. We define $r' := \frac{1}{r_1}$. Clearly $r'r < 1$, and so

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left|b_k\frac{x^{mk}}{m}\right| \leq C\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{m}(r')^k r^{mk} = C\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{m}(r'r^m)^k$$
$$= C\sum_{m=1}^{\infty}\frac{1}{m}\frac{r'r^m}{1-r'r^m} < \infty.$$

Step 3 Holomorphicity of $\mathbb{E}\left[W^\infty(f)(x)\right]$:

$$\left|\log\left(\prod_{k=1}^{\infty}\frac{1}{(1-x^k)^{b_k}}\right)\right| \leq \sum_{k=1}^{\infty}|b_k||\log(1-x^k)| \leq \beta\sum_{k=1}^{\infty}|b_k||x^k| < \infty$$

Step 4 Existence of the moments. Let $\sigma \in \mathcal{C}_\lambda$. We have

$$\begin{aligned}\Big(W^{1,n}(f_1)(x)W^n(f_2)(x)\Big)(\sigma) &= \left(\prod_{m=1}^{l(\lambda)} f_1(\theta_m x^{\lambda_m})\right)\left(\prod_{m=1}^{l(\lambda)} f_2(\theta_m x^{\lambda_m})\right)\\ &= \prod_{m=1}^{l(\lambda)} \Big(f_1(\theta_m x^{\lambda_m}) f_2(\theta_m x^{\lambda_m})\Big)\\ &= \prod_{m=1}^{l(\lambda)} (f_1 f_2)(\theta_m x^{\lambda_m}) = \left(W^{1,n}(f_1 f_2)(x)\right)(\sigma),\end{aligned}$$

and step 4 now follows from steps 1, 2, and 3.

□

5.8.4 Convergence against the limit

We give in this section two proofs of theorem 5.8.1. The idea of the first proof is to show $W^{1,n}(f)(x) \xrightarrow{d} W^{1,\infty}(f)(x)$ and then use uniform integrability. The idea of the second is to show that $\mathbb{E}\left[W^{1,n}(f)(x)\right] \to \mathbb{E}\left[W^{1,\infty}(f)(x)\right]$ for $x \in [0, r']$ with r' small enough and then to apply the theorem of Montel.

Note that the second proof does not imply $W^{1,n}(f)(x) \xrightarrow{d} W^{1,\infty}(f)(x)$. One would need that $W^{1,n}(f)(x)$ is uniquely defined by its moments to apply theorem 2.5.16. Unfortunately we haven't been able to prove or disprove this.

First proof of theorem 5.8.1 We first prove

Lemma 5.8.8. *Choose an $0 < u$ $(\le r)$ such that $f(x) \neq 0$ for $x \in B_u(0)$. We have for all fixed $x \in B_u(0)$*

$$\sum_{m=1}^{n} C_m^{(n)} \log\big(f(x^m)\big) \xrightarrow{d} \sum_{m=1}^{\infty} Y_m \log\big(f(x^m)\big) \qquad (n \to \infty) \tag{5.8.14}$$

$$W^{1,n}(f)(x) \xrightarrow{d} W^{1,\infty}(f)(x) \qquad (n \to \infty) \tag{5.8.15}$$

Remark: While the function $\sum_{m=1}^{n} C_m^{(n)} \log\big(f(x^m)\big)$ is not guaranteed to be holomorphic in x, it is well-defined with the convention $\log(-y) := \log(y) + i\pi$.

Proof. Since the exponential map is continuous, the second part follows immediately from the first part (a proof of this fact can be found in [6].) We know from theorem 2.1.12 that

$$\mathbb{E}\left[\left|C_m^{(n)} - Y_m\right|\right] \le \frac{2}{n+1}. \tag{5.8.16}$$

We use δ, β and h as in the proof of lemma 5.8.5 to get

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{m=1}^{n} (Y_m - C_m^{(n)}) \log\big(f(x^m)\big)\right|\right] &\le \sum_{m=1}^{n} \mathbb{E}\left[|Y_m - C_m^{(n)}|\right] |\log\big(f(x^m)\big)|\\ &\le \sum_{m=1}^{n} \frac{2}{n+1} |\log\big(1 + x^m h(x^m)\big)|\\ &\le \frac{2}{n+1}\left(C + \sum_{m=k_0}^{n} \beta(1+\delta) r^m\right) \longrightarrow 0 \quad (n \to \infty).\end{aligned}$$

□

Weak convergence does not automatically imply convergence of the expectation. One need some additional properties. We introduce therefore

Definition 5.8.9. *A sequence of (complex valued) random variables $(X_m)_{m\in\mathbb{N}}$ is called uniformly integrable if*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[|X_n|\mathbf{1}_{|X_n|>c}\right] \longrightarrow 0 \text{ for } c \to \infty.$$

One can now use

Lemma 5.8.10 (see [16]). *Let $(X_m)_{m\in\mathbb{N}}$ be uniformly integrable and assume that $X_n \xrightarrow{d} X$. Then,*

$$\mathbb{E}\left[X_n\right] \longrightarrow \mathbb{E}\left[X\right].$$

We now finish the proof of theorem 5.8.1. We define h and δ as in the proof of lemma 5.8.5 and get

$$|W^{1,\infty}(f)(x)| = \prod_{m=1}^{\infty} |f(x^m)|^{Y_m} \leq \prod_{m=1}^{\infty} (1+\delta r^m)^{Y_m} =: \widetilde{F}(r).$$

It is possible that $C_m^{(n)} = Y_m + 1$ and so $\widetilde{F}(r)$ is not automatically an upper bound for $W^{1,n}(f)$, but $F(r) := \prod_{m=1}^{\infty}(1+\delta r^m)^{Y_m+1}$ is. We now have $|W^{1,n}(f)| \leq F(r)$ and $\mathbb{E}\left[|W^{1,n}(f)|\right] \leq \mathbb{E}\left[F(r)\right]$. We get with this $F(r)$

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\left|W^{1,n}(f)(x)\right| \mathbf{1}_{|W^{1,n}(f)(x)|>c}\right] \leq \mathbb{E}\left[|F(r)|\,\mathbf{1}_{F(r)>c}\right] \longrightarrow 0 \qquad (c\to\infty)$$

The sequence $\left(W^{1,n}(f)(x)\right)_{n\in\mathbb{N}}$ is therefore uniformly integrable. Lemmas 5.8.8 and 5.8.10 together prove theorem 5.8.1 for $x \in B_u(0)$ with u as in lemma 5.8.8. $\mathbb{E}\left[W^{1,n}(f)(x)\right]$ and $\mathbb{E}\left[W^{1,\infty}(f)(x)\right]$ are holomorphic in $B_r(0)$ and bounded by $\mathbb{E}\left[F(r)\right]$. Therefore theorem 5.8.1 is true for all $x \in B_r(0)$. □

Second proof of theorem 5.8.1 We first prove a special case.

Lemma 5.8.11. *Assume that $b_k \in \mathbb{R}$ for $1 \leq k < \infty$ and choose $0 < r' < r$ such that either $f(x) \leq 1$ on the interval $x \in [0, r']$ or $f(x) \geq 1$. We then have for $x \in [0, r']$*

$$\mathbb{E}\left[W^{1,n}(f)(x)\right] \to \mathbb{E}\left[W^{1,\infty}(f)(x)\right].$$

Remark: We can choose such an r' because f is holomorphic.

Proof.

Case $f(x) \leq 1$ for $x \in [0, r']$

Inequality $\leq$ Choose $m_0 \in \mathbb{N}$ arbitrary. We have for $n \geq m_0$

$$\mathbb{E}\left[\prod_{m=1}^{n} f(x^m)^{C_m^{(n)}}\right] \leq \mathbb{E}\left[\prod_{m=1}^{m_0} f(x^m)^{C_m^{(n)}}\right]$$

We know that $(C_1^{(n)}, \cdots, C_{m_0}^{(n)}) \xrightarrow{d} (Y_1, \cdots, Y_{m_0})$ and $\prod_{m=1}^{m_0} f(x^m)^{C_m} \leq 1$. This is enough to give

$$\mathbb{E}\left[\prod_{m=1}^{m_0} f(x^m)^{C_m^{(n)}}\right] \to \mathbb{E}\left[\prod_{m=1}^{m_0} f(x^m)^{Y_m}\right] \text{ for } n \to \infty.$$

Therefore,

$$\limsup_{n\to\infty} \mathbb{E}\left[\prod_{m=1}^{n} f(x^m)^{C_m^{(n)}}\right] \leq \inf_{m_0\in\mathbb{N}} \mathbb{E}\left[\prod_{m=1}^{m_0} f(x^m)^{Y_m}\right] = \mathbb{E}\left[W^\infty(f)(x)\right]$$

Inequality $\geq$ We need in this case the Feller coupling. If we would have always $C_m^{(n)} \leq Y_m$ then we would have no problem. We have in fact $C_m^{(n)} \leq Y_m$ if $\xi_{n+1} = 1$. We use therefore a small trick. Remember the definition $B_k = B_k^{(n)} = \{\xi_{n-k} \cdots \xi_{n+1} = 100\cdots 0\}$. We write

$$\begin{aligned}
W^{1,n}(f)(x) &= \prod_{m=1}^{n} f(x^m)^{C_m^{(n)}} = \left(\sum_{k=0}^{n} \mathbf{1}_{B_k}\right) \prod_{m=1}^{n} f(x^m)^{C_m^{(n)}} \\
&= \prod_{m=1}^{n} f(x^m)^{C_m^{(n)}} \mathbf{1}_{B_0} + \sum_{k=1}^{n} f(x^k) \prod_{m=1}^{n} f(x^m)^{C_m^{(n-k)}} \mathbf{1}_{B_k} \\
&= W^{1,n}(f)(x)\mathbf{1}_{B_0} + \sum_{k=1}^{n} f(x^k) W^{1,n-k}(f)(x)\mathbf{1}_{B_k}.
\end{aligned}$$

Let $0 < \epsilon < 1$ be arbitrary and fixed. Since $f(0) = 1$ there exists a k_0 such that $1 - \epsilon < f(x^k) \leq 1$ for $k \geq k_0$. We get

$$\begin{aligned}
\mathbb{E}\left[W^{1,n}(f)\right] &= \mathbb{E}\left[W^{1,n}(f)(x)\mathbf{1}_{B_0}\right] + \mathbb{E}\left[\sum_{k=1}^{n} f(x^k) W^{1,n-k}(f)(x)\mathbf{1}_{B_k}\right] \\
&\geq \mathbb{E}\left[W^{1,\infty}(f)(x)\mathbf{1}_{B_0}\right] + \mathbb{E}\left[\sum_{k=1}^{n} f(x^k) W^{1,\infty}(f)(x)\mathbf{1}_{B_k}\right] \\
&\geq \mathbb{E}\left[W^{1,\infty}(f)(x)\mathbf{1}_{B_0}\right] + \mathbb{E}\left[\sum_{k=1}^{k_0} f(x^k) W^{1,\infty}(f)(x)\mathbf{1}_{B_k}\right] \\
&\quad + \mathbb{E}\left[\sum_{k=k_0+1}^{n} (1-\epsilon) W^{1,\infty}(f)(x)\mathbf{1}_{B_k}\right].
\end{aligned}$$

We have that $\mathbb{P}\left[B_k\right] = \frac{1}{n+1}$, $f(x^k)$ is bounded for $1 \leq k \leq k_0$ and all moments of $W^{1,\infty}(f)$ exist. We can now apply the Schwartz inequality (for L^2) to see that the first two summands go to 0. We can replace k_0 by 0 in the third summand by the same argument.

Case $f(x) \geq 1$ **for** $x \in [0, r']$ The arguments are almost the same. We have to exchange all $\leq$ and $\geq$ and check that

$$\mathbb{E}\left[\prod_{m=1}^{m_0} f(x^m)^{C_m^{(n)}}\right] \longrightarrow \mathbb{E}\left[\prod_{m=1}^{m_0} f(x^m)^{Y_m}\right] \tag{5.8.17}$$

Since $C_m^{(n)} \leq Y_m + 1$, we have a common (integrable) upper bound. We also know from theorem 2.1.12 that

$$\mathbb{P}\left[(C_1^{(n)}, \cdots, C_{m_0}^{(n)}) \neq (Y_1, \cdots, Y_{m_0})\right] \to 0 \ (n \to \infty)$$

These two facts together prove (5.8.17).

$\square$

We now extend lemma 5.8.11 to arbitrary x and b_k. We have constructed in the proof of lemma 5.8.11 an lower and an upper bound for $\mathbb{E}\left[W^{1,n}(f)(x)\right]$ and showed that they converge to the same limit as $n \to \infty$. One could try to modify this proof to apply it to general x and b_k, but this is rather difficult. It is easier to use the theorem of Montel (see [15]).

Lemma 5.8.12. *We have for any holomorphic function f and $x \in B_r(0)$*

$$\mathbb{E}\left[W^{1,n}(f)(x)\right] \to \mathbb{E}\left[W^{\infty}(f)(x)\right] \text{ for } n \to \infty.$$

Proof.

Step 1 We show first that lemma 5.8.11 is true for arbitrary $x \in B_r(0)$ and all $b_k \in \mathbb{R}$ (i.e. with no condition that either $f(x) \geq 1$ or $f(x) \leq 1$ on a whole interval). We use the $F(r)$ from the first proof. We apply the theorem of Montel with the upper bound $\mathbb{E}\left[F(r)\right]$ for $\mathbb{E}\left[W^{1,n}(f)(x)\right]$.
Suppose that there exists a $x_0 \in B_r(0)$ where $\mathbb{E}\left[W^{1,n}(f)(x_0)\right] \nrightarrow \mathbb{E}\left[W^{1,\infty}(f)(x_0)\right]$. Then there exists a $\epsilon > 0$ and a sequence $\Lambda \subset \mathbb{N}$ with $|\mathbb{E}\left[W^{1,n}(f)(x_0)\right] - \mathbb{E}\left[W^{1,\infty}(f)(x_0)\right]| > \epsilon$ for $n \in \Lambda$. We apply the theorem of Montel and get a subsequence $\Lambda' \subset \Lambda$ and a holomorphic function h with

$$\mathbb{E}\left[W^{1,n}(f)(x)\right] \longrightarrow h(x) \text{ for } x \in B_r(0) \text{ and } n \in \Lambda'.$$

We know form lemma 5.8.11 that h has to agree with $\mathbb{E}\left[W^{1,\infty}(f)(x)\right]$ on $[0, r']$. This is a contradiction, since $\mathbb{E}\left[W^{1,\infty}(f)(x_0)\right] \neq h(x_0)$.

Step 2 We now prove the lemma. Let b_k be arbitrary. We define $b_k(s) := \mathrm{Re}(b_k) + s\mathrm{Im}(b_k)$ and

$$f^{(s)}(x) := 1 + \sum_{k=1}^{\infty} b_k(s) x^k.$$

It follows as in lemmas 5.8.5 and 5.8.7 that $W^{1,\infty}\left(f^{(s)}\right)(x)$ and $\mathbb{E}\left[W^{1,\infty}\left(f^{(s)}\right)(x)\right]$ are holomorphic functions in x and s. We know from step 1 that

$$\mathbb{E}\left[W^{1,n}\left(f^{(s)}\right)(x)\right] \longrightarrow \mathbb{E}\left[W^{1,\infty}\left(f^{(s)}\right)(x)\right] \qquad \text{for } x \in B_r(0), s \in \mathbb{R}.$$

We use as in step 1 the theorem of Montel and get

$$\mathbb{E}\left[W^{1,n}\left(f^{(s)}\right)(x)\right] \longrightarrow \mathbb{E}\left[W^{1,\infty}\left(f^{(s)}\right)(x)\right] \qquad \text{for } x \in B_r(0), s \in \mathbb{C}.$$

We now put $s = i$ and are done.

□

Putting lemmas 5.8.7 and 5.8.12 together, we have proved that for any holomorphic function f and $x \in B_r(0)$

$$\mathbb{E}\left[W^{1,n}(f)(x)\right] \to \mathbb{E}\left[W^{1,\infty}(f)(x)\right] = \prod_{k=1}^{\infty} \frac{1}{(1-x^k)^{b_k}},$$

i.e. the special case of theorem 5.8.1 in just one variable and when $b_{0,0} = 1$. By lemma 5.8.2, this is enough to prove the full theorem (in one variable).

5.8.5 Proof of theorem 5.8.1 for W^2

The proof of theorem 5.8.1 for W^2 is almost the same. One only has to replace $W^{1,\infty}(f)(x)$ with

$$W^{2,\infty}(f)(x) := \prod_{m=1}^{\infty} f(x^m, \alpha_1^m, \alpha_2^m, \cdots)^{Y_m}. \tag{5.8.18}$$

Chapter 6

Characteristic polynomial and associated class functions on other groups

---

We can also use the techniques of section 5 for some other groups than S_n. These are the alternating group and the Weyl groups of classical groups. We do not give here the definition of a Weyl group, since this would go to far and can be found in many books about Lie groups, for instance in [9]. We will only give a presentation of the group and write down the generating functions. The asymptotic behavior follows directly form theorem 5.5.1 and theorem 5.8.1.

6.1 The alternating group $\mathcal{A}_n$

It is natural to ask if we can use the techniques of section 5 to obtain generating functions for subgroups of S_n. Section 5 is based on (2.6.14) in lemma 2.6.8. Since this formula is only true for class functions on S_n, the possible subgroups have to be normal. Therefore the only candidate is the alternating group $\mathcal{A}_n$.

6.1.1 Definitions

Definition 6.1.1. *A $\sigma \in S_n$ is called* even *if σ can be written as an even number of transpositions. Otherwise σ is called* odd*. The* alternating group *$\mathcal{A}_n$ is the subset of S_n of all even permutations. The signature $\epsilon(\sigma)$ of a permutation is 1 for even permutations, -1 for odd ones.*

Lemma 6.1.2. *The signature ϵ is a group homomorphism and $\mathcal{A}_n = ker(\epsilon)$.*

Definition 6.1.3. *We write $\mathbb{E}_{\mathcal{A}_n}[f]$ for the expectation with respect to the Haar-measure $\mu_{\mathcal{A}_n}$ on $\mathcal{A}_n$. Explicitly we have for $n \geq 2$ (only, because $\mathcal{S}_1 = \mathcal{A}_1 = \{1\}$)*

$$\mathbb{E}_{\mathcal{A}_n}[f] = \frac{2}{n!} \sum_{\sigma \in \mathcal{A}_n} f(\sigma). \tag{6.1.1}$$

6.1.2 Generating functions for W^1 and W^2 on $\mathcal{A}_n$

We prove in this subsection

Theorem 6.1.4. *Let θ and ϑ be random variables with values in S^1 and P be a polynomial with*

$$P(x_1, x_2) = \sum_{k_1,k_2=0}^{\infty} b_{k_1,k_2} x_1^{k_1} x_2^{k_2}.$$

We define as in (5.3.1) $\alpha_{k_1,k_2} := \mathbb{E}\left[\theta^{k_1}\vartheta^{k_2}\right]$. *We have for* $n \geq 2$

$$\mathbb{E}_{\mathcal{A}_n}\left[W^1(P)(x_1,x_2)\right] = \left[\prod_{k_1,k_2=0}^{\infty}(1 - x_1^{k_1}x_2^{k_2}t)^{-b_{k_1,k_2}\alpha_{k_1,k_2}}\right]_n + \left[\prod_{k_1,k_2=0}^{\infty}(1 + x_1^{k_1}x_2^{k_2}t)^{-b_{k_1,k_2}\alpha_{k_1,k_2}}\right]_n \tag{6.1.2}$$

$$\mathbb{E}_{\mathcal{A}_n}\left[W^2(P)(x_1,x_2)\right] = \left[\prod_{k_1,k_2=0}^{\infty}(1 - \alpha_{k_1,k_2}x_1^{k_1}x_2^{k_2}t)^{-b_{k_1,k_2}}\right]_n + \left[\prod_{k_1,k_2=0}^{\infty}(1 + \alpha_{k_1,k_2}x_1^{k_1}x_2^{k_2}t)^{-b_{k_1,k_2}}\right]_n \tag{6.1.3}$$

The products in (6.1.2), (6.1.3) are holomorphic for $|t| < 1$ *and* $\max\{|x_1|,|x_2|\} \leq 1$.

The idea of the proof is to reformulate $\mathbb{E}_{\mathcal{A}_n}[..]$ and to use the results of section 5. We start with

Lemma 6.1.5. *We have for each* $f : S_n \to \mathbb{C}$

$$\mathbb{E}_{\mathcal{A}_n}\left[f|_{\mathcal{A}_n}\right] = \mathbb{E}_{\mathcal{S}_n}\left[f\right] + \mathbb{E}_{\mathcal{S}_n}\left[\epsilon \cdot f\right] \text{ for } n \geq 2 \tag{6.1.4}$$

and $\epsilon(\sigma) = \prod_{m=1}^{l(\lambda)}(-1)^{\lambda_m+1}$ *for* $\sigma \in \mathcal{C}_\lambda$.

Proof. We have for $n \geq 2$

$$\mathbb{E}_{\mathcal{A}_n}\left[f|_{\mathcal{A}_n}\right] = \frac{2}{n!}\sum_{\substack{\sigma\in S_n\\ \epsilon(\sigma)=1}} f(\sigma) = \frac{1}{n!}\sum_{\sigma\in S_n}\big((1+\epsilon)f\big)(\sigma) = \mathbb{E}_{\mathcal{S}_n}\left[f\right] + \mathbb{E}_{\mathcal{S}_n}\left[\epsilon\cdot f\right]$$

This proves (6.1.4). The second statement is trivial. □

We can now prove theorem 6.1.4:

Proof of theorem 6.1.4. We calculate $\sum_{n=0}^{\infty}\mathbb{E}\left[\epsilon W^{1,n}(P)\right]t^n$ and use lemma 6.1.5. This calculation is very similar to the calculations in the proof of theorem 5.3.1. We therefore simplify the proof by assuming $\theta \equiv \vartheta \equiv 1$ and that P is only dependent on one variable. We get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{E}\left[\epsilon W^{1,n}(P)(x)\right]t^n &= \sum_{\lambda}\frac{1}{z_\lambda}\prod_{m=1}^{l(\lambda)}(-1)^{\lambda_m+1}P(x^{\lambda_m})t^{|\lambda|} = \exp\left(\sum_{m=1}^{\infty}\frac{1}{m}(-1)^{m+1}P(x^m)t^m\right)\\ &= \exp\left(-\sum_{m=1}^{\infty}\frac{1}{m}(-1)^m\sum_{k=0}^{\infty}b_kx^{km}t^m\right) = \exp\left(-\sum_{k=0}^{\infty}b_k\log(1+x^kt)\right)\\ &= \prod_{k=0}^{\infty}(1+x^kt)^{-b_k}\end{aligned}$$

□

6.2 The Weyl group of $SO(2n)$

Let D be the set of diagonal matrices with diagonal entries $1, -1$. The Weyl group $\mathcal{W}_n$ of $SO(2n)$ is equal to DS_n. We define $Z_{\mathcal{W}_n}(x)(w) := \det(I - xw)$ for $w \in \mathcal{W}_n$. We have to take a closer look at $\mu_{\mathcal{W}_n}$ to write down a generating function for the moments of $Z_{\mathcal{W}_n}(x)$. Any element w of $\mathcal{W}_n$ can be written uniquely as $w = dg$ with $d \in D, g \in S_n$. Therefore $\mu_{\mathcal{W}_n} = \mu_D \times \mu_{S_n}$ and the diagonal matrices are independent of S_n. A simple calculation shows that the diagonal entries d_i in D are iid with $\mathbb{P}[d_i = 1] = \mathbb{P}[d_i = -1] = \frac{1}{2}$. This is precisely the definition of W^2 in (5.2.4). We use theorem 5.3.1 and the example in section 5.4.3 to get

$$\mathbb{E}\left[Z^{s_1}_{\mathcal{W}_n}(x_1) Z^{s_2}_{\mathcal{W}_n}(x_2)\right] = \left[\prod_{\substack{k_1,k_2=0\\2|(k_1-k_2)}}^{\infty} (1 - x_1^{k_1} x_2^{k_2} t)^{\binom{s_1}{k_1}\binom{s_2}{k_2}(-1)^{k_1+k_2+1}}\right]_n$$

6.3 The Weyl group of $SO(2n+1)$

Let D be as above. The Weyl group $\mathcal{W}'_n$ of $SO(2n+1)$ is equal to $D\mathcal{A}_n$. We can argue as above and get with theorem 6.1.4

$$\begin{aligned}\mathbb{E}\left[Z^{s_1}_{\mathcal{W}'_n}(x_1) Z^{s_2}_{\mathcal{W}'_n}(x_2)\right] &= \left[\prod_{\substack{k_1,k_2=0\\2|(k_1-k_2)}}^{\infty} (1 - x_1^{k_1} x_2^{k_2} t)^{\binom{s_1}{k_1}\binom{s_2}{k_2}(-1)^{k_1+k_2+1}}\right]_n \\ &+ \left[\prod_{\substack{k_1,k_2=0\\2|(k_1-k_2)}}^{\infty} (1 + x_1^{k_1} x_2^{k_2} t)^{\binom{s_1}{k_1}\binom{s_2}{k_2}(-1)^{k_1+k_2+1}}\right]_n\end{aligned}$$

6.4 The Weyl group of $SU(n)$

The Weyl group $\widetilde{\mathcal{W}_n}$ of $SU(n)$ is equal to S_n, shrunk to the subspace $\mathcal{T} = \{(x_1, \cdots, x_n) \in \mathbb{C}^n; \sum x_i = 0\}$. It is easy to see that $\mathbb{C}^n \cong \mathcal{T} \oplus \mathbb{C}(1, 1, \cdots, 1)$ and the action of S_n on $\mathbb{C}(1, 1, \cdots, 1)$ is trivial. Thus

$$\mathbb{E}\left[Z^{s_1}_{\widetilde{\mathcal{W}_n}}(x_1) Z^{s_2}_{\widetilde{\mathcal{W}_n}}(x_2)\right] = \left[\frac{\prod_{k_1=0}^{s_1}\prod_{k_2=0}^{s_2}\left(1 - x_1^{k_1} x_2^{k_2} t\right)^{\binom{s_1}{k_1}\binom{s_2}{k_2}(-1)^{k_1+k_2+1}}}{(1-x_1)^{s_1}(1-x_2)^{s_2}}\right]_n$$

Bibliography

[1] The 10^{20}-th zero of the riemann zeta function and 175 million of its neighbors. 1992. 3

[2] Lars V. Ahlfors. *Complex analysis.* McGraw-Hill Book Co., New York, third edition, 1978. An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics. 9

[3] T. Apostol. *Introduction to analytic number theory.* Springer-Verlag, New York, 1984. 35, 69

[4] Tom M. Apostol. *Modular functions and Dirichlet series in number theory*, volume 41 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1990. 22

[5] Richard Arratia, A.D. Barbour, and Simon Tavaré. *Logarithmic combinatorial structures: a probabilistic approach.* EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2003. 6, 7, 8

[6] Patrick Billingsley. *Convergence of probability measures.* Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication. 15, 18, 74

[7] B.Riemann. Über die anzahl der primzahlen unter einer gegebenen grösse. pages 671–680, 1859. 2

[8] T. Bröcker and T. tom Dieck. *Representations of Compact Lie groups*, volume 98 of *Graduate Texts in Mathematics.* pub-SV, pub-SV:adr, 1985. 1

[9] Daniel Bump. *Lie groups*, volume 225 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 2004. 6, 22, 25, 37, 42, 79

[10] Ovidiu Costin and Joel L. Lebowitz. Gaussian fluctuation in random matrices. *Phys. Rev. Lett.*, 75(1):69–72, Jul 1995. 3

[11] Paul-Olivier Dehaye and Zeindler Dirk. On averages of randomized class functions on the symmetric groups and their asymptotics. 59

[12] H.Kösters F. Götze. On the second-order correlation function of the characteristic polynomial of a hermitian wigner matrix. *Communications in Mathematical Physics*, pages 1183–1205, 2009. 2

[13] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics.* Cambridge University Press, New York, NY, USA, 2009. 20, 21

[14] Eberhard Freitag and Rolf Busam. *Complex analysis.* Universitext. Springer-Verlag, Berlin, 2005. Translated from the 2005 German edition by Dan Fulea. 9, 20, 55, 56, 68

[15] Klaus Fritzsche and Hans Grauert. *From holomorphic functions to complex manifolds*, volume 213 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 2002. 77

[16] Allan Gut. *Probability: a graduate course.* Springer Texts in Statistics. Springer, New York, 2005. 15, 18, 75

[17] B.M. Hambly, P. Keevash, N. O'Connell, and D. Stark. The characteristic polynomial of a random permutation matrix. *Stochastic Process. Appl.*, 90(2):335–346, 2000. 3, 12, 27, 31, 53

[18] Christopher Hughes. *On the Characteristic Polynomial of a Random Unitary Matrix and the Riemann Zeta Function.* PhD thesis, University of Bristol, 2001. 3

[19] J.P. Keating and N.C. Snaith. Random matrix theory and $\zeta(1/2+it)$. *Commun. Math. Phys.*, 214:57–89, 2000. 2, 57

[20] L. Kuipers and H. Niederreiter. *Uniform Distribution of Sequences.* Wiley, New-York, 1974. 9, 10, 11, 15

[21] A. Laurincikas. *Limit Theorems for the Riemann Zeta Function.* Kluwer, 1996. 2

[22] I. G. Macdonald. *Symmetric functions and Hall polynomials.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications. 5, 6, 22

[23] Steven J. Miller and Ramin Takloo-Bighash. *An invitation to modern number theory.* Princeton University Press, Princeton, NJ, 2006. With a foreword by Peter Sarnak. 1

[24] Bruce E. Sagan. *The symmetric group*, volume 203 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 2001. Representations, combinatorial algorithms, and symmetricfunctions. 22

[25] Kelly Wieand. Eigenvalue distributions of random permutation matrices. *Ann. Probab.*, 28(4):1563–1587, 2000. 3

[26] Kelly Wieand. Permutation matrices, wreath products, and the distribution of eigenvalues. *J. Theoret. Probab.*, 16(3):599–623, 2003. 3, 64

[27] D. Zeindler. Permutation matrices and the moments of their characteristics polynomials. *Electronic Journal of Probability*, 15:1092–1118, 2010. 37